

**Form 5 Chapter 3**  
**Integration**  
**Fully-Worked Solutions**

**UPSKILL 3.2a**

$$1 \text{ (a) } \int -10 \, dx$$

$$= -10x + c$$

$$\text{(b) } \int 0 \, dx$$

$$= 0x + c$$

$$= c$$

$$\text{(c) } \int dx$$

$$= 1x + c$$

$$= x + c$$

$$\text{(d) } \int 6x \, dx$$

$$= \frac{6x^2}{2} + c$$

$$= 3x^2 + c$$

$$\text{(e) } \int 6x^2 \, dx$$

$$= 6 \left( \frac{x^3}{3} \right) + c$$

$$= 2x^3 + c$$

$$\text{(f) } \int \frac{5}{3} x^4 \, dx$$

$$= \frac{5}{3} \left( \frac{x^5}{5} \right) + c$$

$$= \frac{x^5}{3} + c$$

$$\text{(g) } \int \frac{6}{x^4} \, dx$$

$$= \int 6x^{-4} \, dx$$

$$= \frac{6x^{-3}}{-3} + c$$

$$= -2x^{-3} + c$$

$$= -\frac{2}{x^3} + c$$

$$\text{(h) } \int -\frac{2}{5x^3} \, dx$$

$$= \int -\frac{2}{5} x^{-3} \, dx$$

$$= -\frac{2}{5} \left( \frac{x^{-2}}{-2} \right) + c$$

$$= \frac{1}{5x^2} + c$$

$$\text{(i) } \int \frac{3}{x^7} \, dx$$

$$= \int 3x^{-7} \, dx$$

$$= 3 \left( \frac{x^{-6}}{-6} \right) + c$$

$$= -\frac{1}{2x^6} + c$$

$$\text{(j) } \int \frac{5}{3x^6} \, dx$$

$$= \int \frac{5}{3} x^{-6} \, dx$$

$$= \frac{5}{3} \frac{x^{-5}}{-5} + c$$

$$= -\frac{1}{3x^5} + c$$

$$2 \text{ (a) } \int (5x^4 + 3x^2 - 4) \, dx$$

$$= \frac{5x^5}{5} + \frac{3x^3}{3} - 4x + c$$

$$= x^5 + x^3 - 4x + c$$

$$\text{(b) } \int \left( \frac{2}{5} x^3 + \frac{1}{4x^3} \right) \, dx$$

$$= \int \left( \frac{2}{5} x^3 + \frac{1}{4} x^{-3} \right) \, dx$$

$$= \frac{2x^4}{20} + \frac{1}{4} \left( \frac{x^{-2}}{-2} \right) + c$$

$$= \frac{x^4}{10} - \frac{1}{8x^2} + c$$

$$\begin{aligned}
 \text{(c)} \quad & \int \left( 3 - \frac{4}{x^2} + \frac{6}{x^3} \right) dx \\
 &= \int (3 - 4x^{-2} + 6x^{-3}) dx \\
 &= 3x - 4 \frac{x^{-1}}{-1} + 6 \left( \frac{x^{-2}}{-2} \right) + c \\
 &= 3x + \frac{4}{x} - \frac{3}{x^2} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad & \int \left( 2 + \frac{4}{x^2} - \frac{3}{x^4} \right) dx \\
 &= \int (2 + 4x^{-2} - 3x^{-4}) dx \\
 &= 2x + \frac{4x^{-1}}{-1} - \frac{3x^{-3}}{-3} + c \\
 &= 2x - \frac{4}{x} + \frac{1}{x^3} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{3} \text{ (a)} \quad & \int 2x^2(4x-6) dx \\
 &= \int (8x^3 - 12x^2) dx \\
 &= \frac{8x^4}{4} - \frac{12x^3}{3} + c \\
 &= 2x^4 - 4x^3 + c
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad & \int (3x+2)(2x-1) dx \\
 &= \int (6x^2 + x - 2) dx \\
 &= \frac{6x^3}{3} + \frac{x^2}{2} - 2x + c \\
 &= 2x^3 + \frac{x^2}{2} - 2x + c
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad & \int (2-3x)^2 dx \\
 &= \int (4-12x+9x^2) dx \\
 &= 4x - \frac{12x^2}{2} + \frac{9x^3}{3} + c \\
 &= 4x - 6x^2 + 3x^3 + c
 \end{aligned}$$

$$\begin{aligned}
 \text{(d)} \quad & \left( 1 + \frac{2}{x} \right) \left( 1 - \frac{2}{x} \right) dx \\
 &= \int \left( 1 - \frac{4}{x^2} \right) dx \\
 &= \int (1 - 4x^{-2}) dx \\
 &= x - \frac{4x^{-1}}{-1} + c \\
 &= x + \frac{4}{x} + c
 \end{aligned}$$

$$\begin{aligned}
 \mathbf{4} \text{ (a)} \quad & \int \left( \frac{x^4+2}{x^2} \right)^2 dx \\
 &= \int \left( \frac{x^8+4x^4+4}{x^4} \right) dx \\
 &= \int (x^4+4+4x^{-4}) dx \\
 &= \frac{x^5}{5} + 4x + \frac{4x^{-3}}{-3} + c \\
 &= \frac{x^5}{5} + 4x - \frac{4}{3x^3} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \quad & \int \frac{(x+4)(x-4)}{x^2} dx \\
 &= \int \left( \frac{x^2-16}{x^2} \right) dx \\
 &= \int (1-16x^{-2}) dx \\
 &= x - \frac{16x^{-1}}{-1} + c \\
 &= x + \frac{16}{x} + c
 \end{aligned}$$

$$\begin{aligned}
 \text{(c)} \quad & \int \frac{3x^3-3x^2+4}{x^2} dx \\
 &= \int (3x-3+4x^{-2}) dx \\
 &= \frac{3x^2}{2} - 3x + \frac{4x^{-1}}{-1} + c \\
 &= \frac{3x^2}{2} - 3x - \frac{4}{x} + c
 \end{aligned}$$

**UPSKILL 3.2b**

$$\begin{aligned} 1 \text{ (a)} \int (2x+1)^4 dx &= \frac{(2x+1)^5}{5(2)} + c \\ &= \frac{(2x+1)^5}{10} + c \end{aligned}$$

$$\begin{aligned} \text{(b)} \int (2-x)^3 dx &= \frac{(2-x)^4}{4(-1)} + c \\ &= -\frac{(2-x)^4}{4} + c \end{aligned}$$

$$\begin{aligned} \text{(c)} \int 4(3x-2)^5 dx &= \frac{4(3x-2)^6}{6(3)} + c \\ &= \frac{2}{9}(3x-2)^6 + c \end{aligned}$$

$$\begin{aligned} \text{(d)} \int 3\left(\frac{1}{2}x-4\right)^3 dx &= 3 \frac{\left(\frac{1}{2}x-4\right)^4}{4\left(\frac{1}{2}\right)} + c \\ &= \frac{3}{2}\left(\frac{1}{2}x-4\right)^4 + c \end{aligned}$$

$$\begin{aligned} \text{(e)} \int 2(3x-2)^{-2} dx &= \frac{2(3x-2)^{-1}}{-1(3)} + c \\ &= -\frac{2}{3}(3x-2)^{-1} + c \end{aligned}$$

$$\begin{aligned} \text{(f)} \int 2(4-3x)^{-6} dx &= \frac{2(4-3x)^{-5}}{-5(-3)} + c \\ &= \frac{2}{15}(4-3x)^{-5} + c \end{aligned}$$

$$\begin{aligned} 2 \text{ (a)} \int \frac{1}{(3x-4)^2} dx &= \int (3x-4)^{-2} dx \\ &= \frac{(3x-4)^{-1}}{-1(3)} + c \\ &= -\frac{1}{3(3x-4)} + c \end{aligned}$$

$$\begin{aligned} \text{(b)} \int \frac{-2}{(2x-3)^4} dx &= \int -2(2x-3)^{-4} dx \\ &= \frac{-2(2x-3)^{-3}}{-3(2)} + c \\ &= \frac{1}{3(2x-3)^3} + c \end{aligned}$$

$$\begin{aligned} \text{(c)} \int \frac{3}{(4x-3)^4} dx &= \int 3(4x-3)^{-4} dx \\ &= \frac{3(4x-3)^{-3}}{-3(4)} + c \\ &= -\frac{1}{4(4x-3)^3} + c \end{aligned}$$

$$\begin{aligned} \text{(d)} \int \frac{5}{(4-3x)^3} dx &= \int 5(4-3x)^{-3} dx \\ &= \frac{5(4-3x)^{-2}}{-2(-3)} + c \\ &= \frac{5}{6(4-3x)^2} + c \end{aligned}$$

$$\begin{aligned} \text{(e)} \int \frac{3}{4(6-x)^5} dx &= \int \frac{3(6-x)^{-5}}{4} dx \\ &= \frac{3(6-x)^{-4}}{4(-4)(-1)} + c \\ &= \frac{3}{16(6-x)^4} + c \end{aligned}$$

$$\begin{aligned}
 \text{(f)} \quad & \int \frac{2}{3(4-2x)^2} dx \\
 &= \int \frac{2(4-2x)^{-2}}{3} dx \\
 &= \frac{2(4-2x)^{-1}}{3(-1)(-2)} \\
 &= \frac{1}{3(4-2x)} + c
 \end{aligned}$$

### UPSKILL 3.2c

1  $\frac{dy}{dx} = x^2(4x-3)$

$$y = \int (4x^3 - 3x^2) dx$$

$$y = \frac{4x^4}{4} - \frac{3x^3}{3} + c$$

$$y = x^4 - x^3 + c$$

Since the curve passes through the point (2, 5), thus  $x = 2$  and  $y = 5$ .

$$5 = 2^4 - 2^3 + c$$

$$c = -3$$

Hence, the equation of the curve is

$$y = x^4 - x^3 - 3.$$

2  $\frac{dy}{dx} = (2x-4) dx$

$$y = \int (2x-4) dx$$

$$y = \frac{2x^2}{2} - 4x + c$$

$$y = x^2 - 4x + c$$

Since the curve passes through the point (3, 0), thus  $x = 3$  and  $y = 0$ .

$$0 = 3^2 - 4(3) + c$$

$$c = 3$$

Hence, the equation of the curve is

$$y = x^2 - 4x + 3.$$

3  $\frac{dy}{dx} = \frac{2}{x^2} + 1$

$$y = \int \left( \frac{2}{x^2} + 1 \right) dx$$

$$y = \int (2x^{-2} + 1) dx$$

$$y = \frac{2x^{-1}}{-1} + x + c$$

$$y = -\frac{2}{x} + x + c$$

Since the curve passes through the point (1, 8), thus  $x = 1$  and  $y = 8$ .

$$8 = -\frac{2}{1} + 1 + c$$

$$c = 9$$

Hence, the equation of the curve is

$$y = -\frac{2}{x} + x + 9.$$

4  $\frac{dy}{dx} = 3x^2(x+1)$

$$y = \int 3x^2(x+1) dx$$

$$y = \int (3x^3 + 3x^2) dx$$

$$y = \frac{3x^4}{4} + \frac{3x^3}{3} + c$$

$$y = \frac{3x^4}{4} + x^3 + c$$

Since the curve passes through the point (1, 2), thus  $x = 1$  and  $y = 2$ .

$$2 = \frac{3(1)^4}{4} + 1^3 + c$$

$$c = \frac{1}{4}$$

Hence, the equation of the curve is

$$y = \frac{3x^4}{4} + x^3 + \frac{1}{4}.$$

5  $\frac{dy}{dx} = \frac{1}{(2x-1)^2}$

$$y = \int \frac{1}{(2x-1)^2} dx$$

$$y = \int (2x-1)^{-2} dx$$

$$y = \frac{(2x-1)^{-1}}{-1(2)} + c$$

$$y = -\frac{1}{2(2x-1)} + c$$

Since the curve passes through the point (2, 1), thus  $x = 2$  and  $y = 1$ .

$$1 = -\frac{1}{2[2(2)-1]} + c$$

$$1 = -\frac{1}{6} + c$$

$$c = \frac{7}{6}$$

Hence, the equation of the curve is

$$y = -\frac{1}{2(2x-1)} + \frac{7}{6}.$$

$$6 \quad \frac{dy}{dx} = 2x - 4$$

$$y = \int (2x - 4) dx$$

$$y = x^2 - 4x + c$$

For minimum value,

$$\frac{dy}{dx} = 0$$

$$2x - 4 = 0$$

$$x = 2$$

The minimum value is 3.

This means that the value of  $y = 3$ .

$$3 = 2^2 - 4(2) + c$$

$$c = 7$$

Hence, the equation of the curve is

$$y = x^2 - 4x + 7.$$

### UPSKILL 3.3a

$$1 \text{ (a) } \int_0^3 (2x^3 - 2x - 1) dx$$

$$= \left[ \frac{2x^4}{4} - \frac{2x^2}{2} - x \right]_0^3$$

$$= \left[ \frac{x^4}{2} - x^2 - x \right]_0^3$$

$$= \frac{3^4}{2} - 3^2 - 3 - 0$$

$$= \frac{57}{2}$$

$$\text{(b) } \int_{-1}^1 \left( 3x^2 + x + \frac{1}{x^2} \right) dx$$

$$= \left[ x^3 + \frac{x^2}{2} - \frac{1}{x} \right]_{-1}^1$$

$$= 1^3 + \frac{1^2}{2} - \frac{1}{1} - \left[ (-1)^3 + \frac{(-1)^2}{2} - \frac{1}{-1} \right]$$

$$= \frac{1}{2} - \frac{1}{2}$$

$$= 0$$

$$\text{(c) } \int_0^2 (3x-1)(x^2+1) dx$$

$$= \int_0^2 (3x^3 + 3x - x^2 - 1) dx$$

$$= \left[ \frac{3x^4}{4} + \frac{3x^2}{2} - \frac{x^3}{3} - x \right]_0^2$$

$$= \frac{3(2)^4}{4} + \frac{3(2)^2}{2} - \frac{2^3}{3} - 2 - 0$$

$$= 13\frac{1}{3}$$

$$\text{(d) } \int_1^2 \frac{3x^3 + 2}{x^3} dx$$

$$= \int_1^2 (3 + 2x^{-3}) dx$$

$$= \left[ 3x + \frac{2x^{-2}}{-2} \right]_1^2$$

$$= \left[ 3x - \frac{1}{x^2} \right]_1^2$$

$$= 3(2) - \frac{1}{2^2} - (3 - 1)$$

$$= \frac{15}{4}$$

$$\text{(e) } \int_0^2 x(x+3)(x-3) dx$$

$$= \int_0^2 x(x^2 - 9) dx$$

$$= \int_0^2 (x^3 - 9x) dx$$

$$= \left[ \frac{x^4}{4} - \frac{9x^2}{2} \right]_0^2$$

$$= \frac{2^4}{4} - \frac{9(2)^2}{2} - 0$$

$$= -14$$

$$\begin{aligned}
\text{(f)} \int_{-1}^1 \left(2x - \frac{1}{x}\right)^2 dx &= \int_{-1}^1 (4x^2 - 4 + x^{-2}) dx \\
&= \left[ \frac{4x^3}{3} - 4x + \frac{x^{-1}}{-1} \right]_{-1}^1 \\
&= \left[ \frac{4x^3}{3} - 4x - \frac{1}{x} \right]_{-1}^1 \\
&= \frac{4}{3} - 4 - 1 - \left( -\frac{4}{3} + 4 + 1 \right) \\
&= -\frac{22}{3}
\end{aligned}$$

$$\begin{aligned}
\text{(g)} \int_0^2 (2x+1)^3 dx &= \left[ \frac{(2x+1)^4}{4(2)} \right]_0^2 \\
&= \left[ \frac{(2x+1)^4}{8} \right]_0^2 \\
&= \frac{5^4}{8} - \frac{1}{8} \\
&= 78
\end{aligned}$$

$$\begin{aligned}
\text{(h)} \int_{-1}^1 \frac{1}{(3-x)^3} dx &= \int_{-1}^1 (3-x)^{-3} dx \\
&= \left[ \frac{(3-x)^{-2}}{-2(-1)} \right]_{-1}^1 \\
&= \left[ \frac{1}{2(3-x)^2} \right]_{-1}^1 \\
&= \frac{1}{2(3-1)^2} - \frac{1}{2(3+1)^2} \\
&= \frac{1}{8} - \frac{1}{32} \\
&= \frac{3}{32}
\end{aligned}$$

$$\begin{aligned}
\text{(i)} \int_0^2 x(x-2)^2 dx &= \int_0^2 x(x^2 - 4x + 4) dx \\
&= \int_0^2 (x^3 - 4x^2 + 4x) dx \\
&= \left[ \frac{x^4}{4} - \frac{4x^3}{3} + \frac{4x^2}{2} \right]_0^2 \\
&= \left[ \frac{x^4}{4} - \frac{4x^3}{3} + 2x^2 \right]_0^2 \\
&= \frac{2^4}{4} - \frac{4(2)^3}{3} + 2(2)^2 - 0 \\
&= \frac{4}{3}
\end{aligned}$$

$$\begin{aligned}
\text{(j)} \int_0^2 \frac{(3x^3 - 2x^5 + 2)}{x^3} dx &= \int_0^2 (3 - 2x^2 + 2x^{-3}) dx \\
&= \left[ 3x - \frac{2x^3}{3} + \frac{2x^{-2}}{-2} \right]_0^2 \\
&= \left[ 3x - \frac{2x^3}{3} - \frac{1}{x^2} \right]_0^2 \\
&= 3(2) - \frac{2}{3}(2)^3 - \frac{1}{2^2} - \left( 3 - \frac{2}{3} - 1 \right) \\
&= \frac{5}{12} - \frac{4}{3} \\
&= -\frac{11}{12}
\end{aligned}$$

$$\begin{aligned}
\mathbf{2(a)} \int_2^6 5f(x) dx &= 5 \int_2^6 f(x) dx \\
&= 5(4) \\
&= 20
\end{aligned}$$

$$\begin{aligned}
\text{(b)} \int_2^6 [5 - f(x)] dx &= \int_2^6 5 dx - \int_2^6 f(x) dx \\
&= [5x]_2^6 - 4 \\
&= 5(6-2) - 4 \\
&= 16
\end{aligned}$$

$$\begin{aligned} \text{(c)} \quad & \int_6^2 f(x) \, dx \\ &= -\int_2^6 f(x) \, dx \\ &= -4 \end{aligned}$$

$$\begin{aligned} \text{(d)} \quad & \int_2^3 f(x) \, dx + \int_3^4 f(x) \, dx + \int_4^6 f(x) \, dx \\ &= \int_2^6 f(x) \, dx \\ &= 4 \end{aligned}$$

$$\begin{aligned} \text{3 (a)} \quad & \int_2^4 f(x) \, dx - \int_7^4 f(x) \, dx \\ &= \int_2^4 f(x) \, dx + \int_4^7 f(x) \, dx \\ &= 6 + 8 \\ &= 14 \end{aligned}$$

$$\begin{aligned} \text{(b)} \quad & \int_2^7 [g(x) + 2x] \, dx \\ &= \int_2^7 g(x) \, dx + \int_2^7 2x \, dx \\ &= 6 + 8 + [x^2]_2^7 \\ &= 14 + 7^2 - 2^2 \\ &= 59 \end{aligned}$$

$$\begin{aligned} \text{(c)} \quad & \int_4^2 3f(x) \, dx + \int_4^7 2g(x) \, dx \\ &= -3\int_2^4 f(x) \, dx + 2\int_4^7 g(x) \, dx \\ &= -3(6) + 2(8) \\ &= -2 \end{aligned}$$

$$\begin{aligned} \text{(d)} \quad & \int_2^4 [g(x) + kx] \, dx = 10 \\ & \int_2^4 g(x) \, dx + \int_2^4 kx \, dx = 10 \\ & 6 + \frac{k}{2} [x^2]_2^4 = 10 \\ & \frac{k}{2} [4^2 - 2^2] = 4 \\ & \frac{k}{2} \times 12 = 4 \\ & k = \frac{2}{3} \end{aligned}$$

$$\begin{aligned} \text{4} \quad & y = \frac{x}{x+4} \\ & \frac{dy}{dx} = \frac{(x+4)(1) - x(1)}{(x+4)^2} \\ & \frac{dy}{dx} = \frac{4}{(x+4)^2} \quad \text{[Shown]} \end{aligned}$$

$$\begin{aligned} y &= \int \frac{4}{(x+4)^2} \, dx \\ 4[y]_2^3 &= \int_2^3 \frac{16}{(x+4)^2} \, dx \\ 4\left[\frac{x}{x+4}\right]_2^3 &= \int_2^3 \frac{16}{(x+4)^2} \, dx \\ \int_2^3 \frac{16}{(x+4)^2} \, dx &= 4\left[\frac{x}{x+4}\right]_2^3 \\ &= 4\left(\frac{3}{7} - \frac{2}{6}\right) \\ &= \frac{8}{21} \end{aligned}$$

$$\begin{aligned} \text{5} \quad & y = \frac{x}{5-x} \\ & \frac{dy}{dx} = \frac{(5-x)(1) - x(-1)}{(5-x)^2} \\ & \frac{dy}{dx} = \frac{5}{(5-x)^2} \quad \text{[Shown]} \end{aligned}$$

$$\begin{aligned} y &= \int \frac{5}{(5-x)^2} \, dx \\ \frac{1}{10}[y]_1^3 &= \frac{1}{10} \int_1^3 \frac{5}{(5-x)^2} \, dx \\ \int_1^3 \frac{1}{2(5-x)^2} \, dx &= \frac{1}{10} \left[\frac{x}{5-x}\right]_1^3 \\ &= \frac{1}{10} \left(\frac{3}{2} - \frac{1}{4}\right) \\ &= \frac{1}{8} \end{aligned}$$

$$6 \quad \frac{dy}{dx} = 4f(x)$$

$$y = 4 \int f(x) dx$$

$$\frac{1}{4}y = \int f(x) dx$$

$$\int_1^2 f(x) dx = \frac{1}{4}[y]_1^2$$

$$= \frac{1}{4} \left[ \frac{3x^2 - 1}{x} \right]_1^2$$

$$= \frac{1}{4} \left[ \frac{3(2)^2 - 1}{2} - \frac{3(1)^2 - 1}{1} \right]$$

$$= \frac{1}{4} \left( \frac{11}{2} - 2 \right)$$

$$= \frac{7}{8}$$

$$7 \quad \frac{d}{dx} \left( \frac{x}{x^2 - 3} \right) = g(x)$$

$$\frac{x}{x^2 - 3} = \int g(x) dx$$

$$\left[ \frac{x}{x^2 - 3} \right]_1^2 = \int_1^2 g(x) dx$$

$$\int_1^2 [2x - g(x)] dx = 2 \left[ \frac{x^2}{2} \right]_1^2 - \left[ \frac{x}{x^2 - 3} \right]_1^2$$

$$= (2^2 - 1^2) - \left( \frac{2}{1} - \frac{1}{-2} \right)$$

$$= 3 - \frac{5}{2}$$

$$= \frac{1}{2}$$

### UPSKILL 3.3b

1 (a) Area of the shaded region

$$= \int_{-1}^4 y dx$$

$$= \int_{-1}^4 (1+x)(4-x) dx$$

$$= \int_{-1}^4 (4-x+4x-x^2) dx$$

$$= \int_{-1}^4 (4+3x-x^2) dx$$

$$= \left[ 4x + \frac{3x^2}{2} - \frac{x^3}{3} \right]_{-1}^4$$

$$= \left[ 16 + \frac{3}{2}(16) - \frac{64}{3} \right] - \left[ -4 + \frac{3}{2} - \left( -\frac{1}{3} \right) \right]$$

$$= \frac{56}{3} - \left( -\frac{13}{6} \right)$$

$$= 20\frac{5}{6} \text{ units}^2$$

(b) Area of the shaded region

$$= \int_1^3 y dx$$

$$= \int_1^3 \left( x + \frac{2}{x^2} \right) dx$$

$$= \int_1^3 (x + 2x^{-2}) dx$$

$$= \left[ \frac{x^2}{2} + 2 \left( \frac{x^{-1}}{-1} \right) \right]_1^3$$

$$= \left[ \frac{x^2}{2} - 2 \left( \frac{1}{x} \right) \right]_1^3$$

$$= \frac{9}{2} - \frac{2}{3} - \left( \frac{1}{2} - 2 \right)$$

$$= 5\frac{1}{3} \text{ units}^2$$

(c) Area of the shaded region

$$= \int_{-1}^2 y dx$$

$$= \int_{-1}^2 (x-1)^2 dx$$

$$= \left[ \frac{(x-1)^3}{3(1)} \right]_{-1}^2$$

$$= \left( \frac{1}{3} - \left( -\frac{8}{3} \right) \right)$$

$$= 3 \text{ units}^2$$

2 (a) Area of the shaded region

$$= \left| \int_{-2}^2 y dx \right|$$

$$= \left| \int_{-2}^2 (x+2)(x-2) dx \right|$$

$$= \left| \int_{-2}^2 (x^2 - 4) dx \right|$$



$$\begin{aligned}
&= \left| \left[ \frac{x^3}{3} - 4x \right]_{-2}^2 \right| \\
&= \left| \frac{8}{3} - 8 - \left( -\frac{8}{3} + 8 \right) \right| \\
&= \left| -10\frac{2}{3} \right| \\
&= 10\frac{2}{3} \text{ units}^2
\end{aligned}$$

(b) Area of the shaded region

$$\begin{aligned}
&= \left| \int_0^3 y \, dx \right| \\
&= \left| \int_0^3 (x-3)^3 \, dx \right| \\
&= \left| \left[ \frac{(x-3)^4}{4(1)} \right]_0^3 \right| \\
&= \left| 0 - \left( \frac{81}{4} \right) \right| \\
&= \frac{81}{4} \text{ units}^2
\end{aligned}$$

**3** (a)  $y = x(x+2)(x-1)$   
 $= x(x^2 + x - 2)$   
 $= x^3 + x^2 - 2x$

Area of the shaded region

$$\begin{aligned}
&= \int_{-2}^0 (x^3 + x^2 - 2x) \, dx + \left| \int_0^1 (x^3 + x^2 - 2x) \, dx \right| \\
&= \left[ \frac{x^4}{4} + \frac{x^3}{3} - x^2 \right]_{-2}^0 + \left| \left[ \frac{x^4}{4} + \frac{x^3}{3} - x^2 \right]_0^1 \right| \\
&= 0 - \left( \frac{16}{4} - \frac{8}{3} - 4 \right) + \left| \left( \frac{1}{4} + \frac{1}{3} - 1 - 0 \right) \right| \\
&= \frac{8}{3} + \left| -\frac{5}{12} \right| \\
&= 3\frac{1}{12} \text{ units}^2
\end{aligned}$$

(b)  $y = x(1+x)(4-x)$   
 $y = x(4 + 3x - x^2)$   
 $y = 4x + 3x^2 - x^3$

Area of the shaded region

$$\begin{aligned}
&= \left| \int_{-1}^0 (4x + 3x^2 - x^3) \, dx \right| + \int_0^4 (4x + 3x^2 - x^3) \, dx \\
&= \left| \left[ 2x^2 + x^3 - \frac{x^4}{4} \right]_{-1}^0 \right| + \left[ 2x^2 + x^3 - \frac{x^4}{4} \right]_0^4 \\
&= \left| 0 - \left( 2 - 1 - \frac{1}{4} \right) \right| + 2(16) + 64 - \frac{4^4}{4} - 0 \\
&= \left| -\frac{3}{4} \right| + 32 \\
&= 32\frac{3}{4} \text{ units}^2
\end{aligned}$$

(c) Area of the shaded region

$$\begin{aligned}
&= \int_0^3 (3+2x-x^2) \, dx + \left| \int_3^4 (3+2x-x^2) \, dx \right| \\
&= \left[ 3x + x^2 - \frac{x^3}{3} \right]_0^3 + \left| \left[ 3x + x^2 - \frac{x^3}{3} \right]_3^4 \right| \\
&= 9 + 9 - 9 + \left| 3(4) + 4^2 - \frac{64}{3} - \left[ 3(3) + 3^2 - \frac{27}{3} \right] \right| \\
&= 9 + \left| \frac{20}{3} - 9 \right| \\
&= 9 + \frac{7}{3} \\
&= 11\frac{1}{3} \text{ units}^2
\end{aligned}$$

**4** (a) Area of the shaded region

$$\begin{aligned}
&= \int_{-1}^3 x \, dy \\
&= \int_{-1}^3 (3y^2 + 2) \, dy \\
&= \left[ \frac{3y^3}{3} + 2y \right]_{-1}^3 \\
&= [y^3 + 2y]_{-1}^3 \\
&= 27 + 6 - (-1 - 2) \\
&= 36 \text{ units}^2
\end{aligned}$$

(b) Area of the shaded region

$$\begin{aligned}
 &= \int_0^3 x \, dy \\
 &= \int_0^3 y^2(3-y) \, dy \\
 &= \int_0^3 (3y^2 - y^3) \, dy \\
 &= \left[ y^3 - \frac{y^4}{4} \right]_0^3 \\
 &= 3^3 - \frac{3^4}{4} \\
 &= 6\frac{3}{4} \text{ units}^2
 \end{aligned}$$

5 (a) Area of the shaded region

$$\begin{aligned}
 &= \left| \int_{-1}^3 x \, dy \right| \\
 &= \left| \int_{-1}^3 (y^2 - 2y - 3) \, dy \right| \\
 &= \left| \left[ \frac{y^3}{3} - y^2 - 3y \right]_{-1}^3 \right| \\
 &= \left| 9 - 9 - 9 - \left( -\frac{1}{3} - 1 + 3 \right) \right| \\
 &= \left| -9 - \frac{5}{3} \right| \\
 &= \left| -10\frac{2}{3} \right| \\
 &= 10\frac{2}{3} \text{ units}^2
 \end{aligned}$$

(b)  $x = y(y+1)(y-1)$

$$\begin{aligned}
 x &= y(y^2 - 1) \\
 x &= (y^3 - y)
 \end{aligned}$$

Area of the shaded region

$$\begin{aligned}
 &= \int_{-1}^0 (y^3 - y) \, dy + \left| \int_0^1 (y^3 - y) \, dy \right| \\
 &= \left[ \frac{y^4}{4} - \frac{y^2}{2} \right]_{-1}^0 + \left| \left[ \frac{y^4}{4} - \frac{y^2}{2} \right]_0^1 \right| \\
 &= 0 - \left( \frac{1}{4} - \frac{1}{2} \right) + \left| \frac{1}{4} - \frac{1}{2} - 0 \right| \\
 &= \frac{1}{4} + \frac{1}{4} \\
 &= \frac{1}{2} \text{ unit}^2
 \end{aligned}$$

6 (a) (i)  $y = -x + 4$  ... (1)

$$y = 4 + 3x - x^2 \quad \dots (2)$$

Substitute (1) into (2) :

$$-x + 4 = 4 + 3x - x^2$$

$$x^2 - 4x = 0$$

$$x(x-4) = 0$$

$$x = 0 \text{ or } x = 4$$

At the turning point,  $x = 0$ ,

$$y = -0 + 4 = 4$$

At the turning point,  $x = 4$ ,

$$y = -4 + 4 = 0$$

Thus, the coordinates of the points  $P$  and  $Q$  are  $(0, 4)$  and  $(4, 0)$  respectively.

(ii) Area of the shaded region

= (Area under the curve from  $x = 0$  to  $x = 4$ ) - (Area of the triangle from  $x = 0$  to  $x = 4$ )

$$= \int_0^4 (4 + 3x - x^2) \, dx - \frac{1}{2} \times 4 \times 4$$

$$= \left[ 4x + \frac{3x^2}{2} - \frac{x^3}{3} \right]_0^4 - 8$$

$$= 4(4) + \frac{3}{2}(4)^2 - \frac{4^3}{3} - 8$$

$$= \frac{56}{3} - 8$$

$$= 10\frac{2}{3} \text{ units}^2$$

(b) (i)  $y = 21$  ... (1)

$$y = 10x - x^2 \quad \dots (2)$$

$$21 = 10x - x^2$$

$$x^2 - 10x + 21 = 0$$

$$(x-3)(x-7) = 0$$

$$x = 3 \text{ or } x = 7$$

Thus, the coordinates of the points  $P$  and  $Q$  are  $(3, 21)$  and  $(7, 21)$  respectively.

(ii) Area of the shaded region

= (Area under the curve from  $x = 3$  to  $x = 7$ ) - (Area of the rectangle from  $x = 3$  to  $x = 7$ )

$$= \int_3^7 (10x - x^2) \, dx - (4 \times 21)$$

$$= \left[ 5x^2 - \frac{x^3}{3} \right]_3^7 - 84$$

$$= \left[ 5(7)^2 - \frac{7^3}{3} - \left( 5(3)^2 - \frac{3^3}{3} \right) \right] - 84$$

$$= \left( \frac{392}{3} - 36 \right) - 84$$

$$= 10 \frac{2}{3} \text{ units}^2$$

(c) (i)  $x = 9 \dots (1)$

$$y^2 = 4x \dots (2)$$

Substitute (1) into (2) :

$$y^2 = 4(9) = 36$$

$$y = \pm 6$$

Thus, the coordinates of the points  $P$  and  $Q$  are  $(9, 6)$  and  $(9, -6)$  respectively.

(ii) Area of the shaded region

= (Area of the rectangle from  $y = -6$  to  $y = 6$ ) - (Area between the curve and the  $y$ -axis from  $y = -6$  to  $y = 6$ )

$$= 12 \times 9 - \int_{-6}^6 \frac{y^2}{4} dy$$

$$= 108 - \left[ \frac{y^3}{12} \right]_{-6}^6$$

$$= 108 - \left[ \frac{6^3}{12} - \left( \frac{-6^3}{12} \right) \right]$$

$$= 108 - (18 + 18)$$

$$= 72 \text{ units}^2$$

(d) (i)  $y = 4x + 1 \dots (1)$

$$y = x^2 + 5 \dots (2)$$

Substitute (2) into (1) :

$$x^2 + 5 = 4x + 1$$

$$x^2 - 4x + 4 = 0$$

$$(x - 2)(x - 2) = 0$$

$$x = 2$$

From (1) :  $y = 4(2) + 1 = 9$

Thus, the coordinates of point  $Q$  are  $(2, 9)$ .

$$y = 4x + 1$$

At the  $y$ -axis,  $x = 0$ .

$$y = 4(0) + 1 = 1$$

Thus, the coordinates of point  $P$  are  $(0, 1)$ .

(ii) Area of the shaded region

= (Area under the curve from  $x = 0$  to  $x = 2$ ) - (Area of trapezium from  $x = 0$  to  $x = 2$ )

$$= \int_0^2 (x^2 + 5) - \frac{1}{2}(1 + 9)(2)$$

$$= \left[ \frac{x^3}{3} + 5x \right]_0^2 - 10$$

$$= \frac{8}{3} + 10 - 10$$

$$= \frac{8}{3} \text{ units}^2$$

7 (a) (i)  $y = x + 3 \dots (1)$

$$y = 2x^2 \dots (2)$$

Substitute (2) into (1) :

$$2x^2 = x + 3$$

$$2x^2 - x - 3 = 0$$

$$x = \frac{3}{2} \text{ or } x = -1$$

$$x = \frac{3}{2} \text{ is not accepted.}$$

$$\therefore x = -1$$

From (1) :  $y = -1 + 3 = 2$

Thus, the coordinates of point  $Q$  are  $(-1, 2)$ .

$$y = x + 3$$

At the  $x$ -axis,  $y = 0$ .

$$y = x + 3$$

$$0 = x + 3$$

$$x = -3$$

Thus, the coordinates of point  $P$  are  $(-3, 0)$ .

(ii) Area of the shaded region

= (Area of the triangle from  $x = -3$  to  $x = -1$ ) + (Area under the curve from  $x = -1$  to  $x = 0$ )

$$= \frac{1}{2} \times 2 \times 2 + \int_{-1}^0 2x^2 dx$$

$$= 2 + \left[ \frac{2x^3}{3} \right]_{-1}^0$$

$$= 2 + \frac{2}{3} [0 - (-1)]$$

$$= 2 + \frac{2}{3}$$

$$= 2 \frac{2}{3} \text{ units}^2$$

(b) (i)  $y = -x + 6 \dots (1)$

$$2x = (y - 2)^2 \dots (2)$$

Substitute (1) into (2) :

$$2x = (-x + 6 - 2)^2$$

$$\begin{aligned}
2x &= (-x+4)^2 \\
2x &= x^2 - 8x + 16 \\
x^2 - 10x + 16 &= 0 \\
(x-2)(x-8) &= 0 \\
x &= 2 \text{ or } x = 8 \\
x = 8 &\text{ is not accepted.} \\
\therefore x &= 2
\end{aligned}$$

From (1):  $y = -2 + 6 = 4$

Thus, the coordinates of point  $Q$  are  $(2, 4)$ .

$$x + y = 6$$

At the  $y$ -axis,  $x = 0$ .

$$0 + y = 6$$

$$y = 6$$

Thus, the coordinates of point  $P$  are  $(0, 6)$ .

(ii) Area of the shaded region  
= (Area between the curve and the  $y$ -axis from  $y = 2$  to  $y = 4$ ) +  
(Area of the triangle from  $y = 4$  to  $y = 6$ )

$$\begin{aligned}
&= \int_2^4 \frac{(y-2)^2}{2} dy + \frac{1}{2} \times 2 \times 2 \\
&= \left[ \frac{(y-2)^3}{2(3)(1)} \right]_2^4 + 2 \\
&= \frac{1}{6} (2^3 - 0) + 2 \\
&= \frac{10}{3} \text{ units}^2
\end{aligned}$$

**8** Area of the shaded region = 81

$$\int_0^k y dx = 81$$

$$\int_0^k 4x^3 dx = 81$$

$$\left[ x^4 \right]_0^k = 81$$

$$k^4 - 0 = 81$$

$$k = 3$$

**9** Area of the shaded region =  $\frac{1}{3}$

$$\int_0^k x dy = \frac{1}{3}$$

$$\int_0^k \frac{y^2}{8} dx = \frac{1}{3}$$

$$\left[ \frac{y^3}{24} \right]_0^k = \frac{1}{3}$$

$$k^3 = \frac{24}{3}$$

$$k^3 = 8$$

$$k = 2$$

**10** Area of the shaded region

$$= \int_0^3 \left[ (-2x^2 + 6x) - (x^2 - 3x) \right] dx$$

$$= \int_0^3 (-3x^2 + 9x) dx$$

$$= \left[ -x^3 + \frac{9x^2}{2} \right]_0^3$$

$$= \left( -3^3 + \frac{9}{2} \times 3^2 - 0 \right)$$

$$= 13 \frac{1}{2} \text{ units}^2$$

**11** Area of the shaded region

$$= \int_{-1}^1 \left[ (x^3 - x) - (x^2 - 1) \right] dx$$

$$= \int_{-1}^1 (x^3 - x^2 - x + 1) dx$$

$$= \left[ \frac{x^4}{4} - \frac{x^3}{3} - x^2 + x \right]_{-1}^1$$

$$= \frac{1}{4} - \frac{1}{3} - 1 + 1 - \left( \frac{1}{4} + \frac{1}{3} - 1 - 1 \right)$$

$$= -\frac{1}{12} - \left( -\frac{17}{12} \right)$$

$$= \frac{4}{3} \text{ units}^2$$

### UPSKILL 3.3c

**1 (a)** Generated volume

$$= \pi \int_1^4 y^2 dx$$

$$= \pi \int_1^4 \left( \frac{4}{x} \right)^2 dx$$

$$= \pi \int_1^4 \frac{16}{x^2} dx$$

$$= \pi \int_1^4 16x^{-2} dx$$

$$= \pi \left[ \frac{16x^{-1}}{-1} \right]_1^4$$

$$= \pi \left[ -\frac{16}{x} \right]_1^4$$

$$= \pi \left[ -\frac{16}{4} - (-16) \right]$$

$$= 12\pi \text{ units}^3$$

(b) Generated volume

$$\begin{aligned} &= \pi \int_1^3 \left(x + \frac{2}{x}\right)^2 dx \\ &= \pi \int_1^3 (x^2 + 4 + 4x^{-2}) dx \\ &= \pi \left[ \frac{x^3}{3} + 4x + \frac{4x^{-1}}{-1} \right]_1^3 \\ &= \pi \left[ \frac{x^3}{3} + 4x - \frac{4}{x} \right]_1^3 \\ &= \pi \left[ 9 + 12 - \frac{4}{3} - \left( \frac{1}{3} + 4 - 4 \right) \right] \\ &= \pi \left( \frac{59}{3} - \frac{1}{3} \right) \\ &= \frac{58}{3} \pi \text{ units}^2 \end{aligned}$$

(c) Generated volume

$$\begin{aligned} &= \pi \int_{-1}^2 (4 - x^2)^2 dx \\ &= \pi \int_{-1}^2 (16 - 8x^2 + x^4) dx \\ &= \pi \left[ 16x - \frac{8x^3}{3} + \frac{x^5}{5} \right]_{-1}^2 \\ &= \pi \left[ 32 - \frac{64}{3} + \frac{32}{5} - \left( -16 + \frac{8}{3} - \frac{1}{5} \right) \right] \\ &= \frac{153}{5} \pi \text{ units}^2 \end{aligned}$$

(d) Generated volume

$$\begin{aligned} &= \pi \int_0^2 (x^2 - 3x)^2 dx \\ &= \pi \int_0^2 (x^4 - 6x^3 + 9x^2) dx \\ &= \pi \left[ \frac{x^5}{5} - \frac{6x^4}{4} + \frac{9x^3}{3} \right]_0^2 \\ &= \pi \left[ \frac{x^5}{5} - \frac{3x^4}{2} + 3x^3 \right]_0^2 \\ &= \pi \left[ \frac{32}{5} - \frac{3(16)}{2} + 3(8) - 0 \right] \\ &= \frac{32}{5} \text{ units}^2 \end{aligned}$$

(e) Generated volume

$$\begin{aligned} &= \pi \int_0^3 [(x-2)^2]^2 dx \\ &= \pi \left[ \frac{(x-2)^5}{5(1)} \right]_0^3 \\ &= \pi \left[ \frac{1}{5} - \left( -\frac{32}{5} \right) \right] \\ &= \frac{33}{5} \pi \text{ units}^2 \end{aligned}$$

(f) Generated volume

$$\begin{aligned} &= \pi \int_{\frac{1}{2}}^1 y^2 dx \\ &= \pi \int_{\frac{1}{2}}^1 \left( 4x - \frac{1}{x} \right)^2 dx \\ &= \pi \int_{\frac{1}{2}}^1 (16x^2 - 8 + x^{-2}) dx \\ &= \pi \left[ \frac{16x^3}{3} - 8x + \frac{x^{-1}}{-1} \right]_{\frac{1}{2}}^1 \\ &= \pi \left[ \frac{16x^3}{3} - 8x - \frac{1}{x} \right]_{\frac{1}{2}}^1 \\ &= \pi \left[ \frac{16}{3} - 8 - 1 - \left[ \left( \frac{16}{3} \right) \left( \frac{1}{8} \right) - \frac{8}{2} - 2 \right] \right] \\ &= \pi \left[ -\frac{11}{3} - \left( -\frac{16}{3} \right) \right] \\ &= \frac{5}{3} \pi \text{ units}^2 \end{aligned}$$

2 (a) Generated volume

$$\begin{aligned} &= \pi \int_0^2 x^2 dy \\ &= \pi \int_0^2 \left( \frac{y^2}{4} \right)^2 dy \\ &= \pi \int_0^2 \frac{y^4}{16} dy \\ &= \pi \left[ \frac{y^5}{16(5)} \right]_0^2 \\ &= \pi \left( \frac{32}{80} - 0 \right) \\ &= \frac{2}{5} \pi \text{ units}^3 \end{aligned}$$

(b) Generated volume

$$\begin{aligned} &= \pi \int_0^1 x^2 dy \\ &= \pi \int_0^1 [(y-1)^2]^2 dy \\ &= \pi \int_0^1 (y-1)^4 dy \\ &= \pi \left[ \frac{(y-1)^5}{5(1)} \right]_0^1 \\ &= \pi \left[ 0 - \left( -\frac{1}{5} \right) \right] \\ &= \frac{1}{5} \pi \text{ units}^3 \end{aligned}$$

(c) Generated volume

$$\begin{aligned} &= \pi \int_0^1 x^2 dy \\ &= \pi \int_0^1 (y^2 - 2y)^2 dy \\ &= \pi \int_0^1 (y^4 - 4y^3 + 4y^2) dy \\ &= \pi \left[ \frac{y^5}{5} - y^4 + \frac{4y^3}{3} \right]_0^1 \\ &= \frac{32}{5} - 16 + \frac{32}{3} - 0 \\ &= \frac{16}{15} \pi \text{ units}^3 \end{aligned}$$

(d) Generated volume

$$\begin{aligned} &= \pi \int_1^2 x^2 dy \\ &= \pi \int_1^2 \left( \frac{y^2}{6} \right)^2 dy \\ &= \pi \int_1^2 \left( \frac{y^4}{36} \right) dy \\ &= \pi \left[ \frac{y^5}{180} \right]_1^2 \\ &= \pi \left[ \frac{2^5}{180} - \frac{1}{180} \right]_1^2 \\ &= \frac{31}{180} \pi \text{ units}^3 \end{aligned}$$

(e) Generated volume

$$\begin{aligned} &= \pi \int_0^9 (9-y) dy \\ &= \pi \left[ 9y - \frac{y^2}{2} \right]_0^9 \\ &= \pi \left( 81 - \frac{81}{2} \right) \\ &= \frac{81}{2} \pi \text{ units}^3 \end{aligned}$$

(f) Generated volume

$$\begin{aligned} &= \pi \int_0^3 x^2 dy \\ &= \pi \int_0^3 [-(y-3)^2]^2 dy \\ &= \pi \int_0^3 [(y-3)^4] dy \\ &= \pi \left[ \frac{(y-3)^5}{5(1)} \right]_0^3 \\ &= \pi \left[ 0 - \left( -\frac{243}{5} \right) \right] \\ &= \frac{243}{5} \pi \text{ units}^3 \end{aligned}$$

3 Generated volume =  $\pi$

$$\begin{aligned} &\pi \int_1^k y^2 dx = \pi \\ &\int_1^k \left( \frac{-\sqrt{2x+1}}{2} \right)^2 dx = 1 \\ &\int_1^k \left( \frac{2x+1}{4} \right) dx = 1 \\ &\int_1^k (2x+1) dx = 4 \\ &[x^2 + x]_1^k = 4 \\ &k^2 + k - (1+1) = 4 \\ &k^2 + k - 6 = 0 \\ &(k+3)(k-2) = 0 \\ &k = -2 \text{ or } k = 2 \\ &k = -2 \text{ is not accepted because } k \text{ has} \\ &\text{to be positive.} \\ &\therefore k = 2 \end{aligned}$$

4 Generated volume =  $\frac{9}{2} \pi$

$$\pi \int_k^4 x^2 dy = \frac{9}{2} \pi$$

$$\int_k^4 (4-y) dy = \frac{9}{2}$$

$$\left[ 4y - \frac{y^2}{2} \right]_k^4 = \frac{9}{2}$$

$$4(4) - \frac{4^2}{2} - \left( 4k - \frac{k^2}{2} \right) = \frac{9}{2}$$

$$8 - 4k + \frac{k^2}{2} - \frac{9}{2} = 0$$

$$16 - 8k + k^2 - 9 = 0$$

$$k^2 - 8k + 7 = 0$$

$$(k-7)(k-1) = 0$$

$$k = 7 \text{ or } k = 1$$

$k = 7$  is not accepted because  $k < 4$ .

$$\therefore k = 1$$

5 Generated volume =  $2\pi$

$$\pi \int_2^k x^2 dy = 2\pi$$

$$\int_2^k (y-2) dy = 2$$

$$\left[ \frac{y^2}{2} - 2y \right]_2^k = 2$$

$$\frac{k^2}{2} - 2k - (2-4) = 2$$

$$\frac{k^2}{2} - 2k + 2 = 2$$

$$k^2 - 4k = 0$$

$$k(k-4) = 0$$

$$k = 0 \text{ or } k = 4$$

$k = 0$  is not accepted.

$$\therefore k = 4$$

6  $y = \frac{8}{x}$

At the turning point,  $y = \frac{4}{3}$ ,

$$\frac{4}{3} = \frac{8}{x}$$

$$4x = 24$$

$$x = 6$$

Generated volume

= (Volume generated by the curve from  $x = 2$  to  $x = 6$ ) - (Volume generated by  $y = \frac{4}{3}$

from  $x = 2$  to  $x = 6$ )

$$= \pi \int_2^6 \left( \frac{8}{x} \right)^2 dx - (\text{Volume of cylinder}$$

generated with a radius of  $\frac{4}{3}$  units and a

height of 4 units)

$$= \pi \int_2^6 64x^{-2} dx - \pi r^2 h$$

$$= \pi \left[ \frac{64x^{-1}}{-1} \right]_2^6 - \pi \left( \frac{4}{3} \right)^2 (4)$$

$$= \pi \left[ -\frac{64}{x} \right]_2^6 + \frac{64}{9} \pi$$

$$= \pi \left[ -\frac{64}{6} - \left( -\frac{64}{2} \right) \right] - \frac{64}{9} \pi$$

$$= \frac{64}{3} \pi - \frac{64}{9} \pi$$

$$= \frac{128}{9} \pi \text{ units}^3$$

7 (a)  $y = 6x - x^2$

$$\frac{dy}{dx} = 6 - 2x$$

At maximum point,

$$\frac{dy}{dx} = 0$$

$$6 - 2x = 0$$

$$x = 3$$

At the turning point  $x = 3$ ,

$$y = 6(3) - 3^2 = 9$$

Thus, the coordinates of the maximum point are  $Q(3, 9)$ .

(b) Generated volume

= (Volume of the cylinder generated by the straight line  $y = 9$  from  $x = 0$  to  $x = 3$ ) - (Volume generated by the curve from  $x = 0$  to  $x = 3$ )

$$= \pi r^2 h - \pi \int_0^3 (6x - x^2)^2 dx$$

$$= \pi (9)^2 (3) - \pi \int_0^3 (36x^2 - 12x^3 + x^4) dx$$

$$= 243\pi - \pi \left[ \frac{36x^3}{3} - \frac{12x^4}{4} + \frac{x^5}{5} \right]_0^3$$

$$\begin{aligned}
&= 243\pi - \pi \left[ \frac{36x^3}{3} - \frac{12x^4}{4} + \frac{x^5}{5} \right]_0^3 \\
&= 243\pi - \pi \left( 12(3)^3 - 3(3)^4 + \frac{3^5}{5} - 0 \right) \\
&= 243\pi - \frac{648}{5}\pi \\
&= \frac{567}{5}\pi \text{ units}^3
\end{aligned}$$

8  $y = \frac{4}{x} \dots (1)$

$y = -2x + 6 \dots (2)$

Substitute (1) into (2) :

$$\frac{4}{x} = -2x + 6$$

$$4 = -2x^2 + 6x$$

$$2x^2 - 6x + 4 = 0$$

$$x^2 - 3x + 2 = 0$$

$$(x-1)(x-2) = 0$$

$$x = 1 \text{ or } x = 2$$

At the turning point,  $x = 1$ ,  $y = \frac{4}{1} = 4$

At the turning point,  $x = 2$ ,  $y = \frac{4}{2} = 2$

Thus, the points of intersection between the straight line and the curve are (1, 4) and (2, 2).

The straight line  $y = -2x + 6$  intersects the  $x$ -axis at (3, 0).

(Volume generated by the curve from  $x = 1$  to  $x = 2$ ) + (Volume of the cone generated by the straight line from  $x = 2$  to  $x = 3$ )

$$\begin{aligned}
&= \pi \int_1^2 \left( \frac{4}{x} \right)^2 dx + \frac{1}{3} \pi r^2 h \\
&= \pi \int_1^2 (16x^{-2}) dx + \frac{1}{3} \pi (2)^2 (3-2) \\
&= \pi \left[ \frac{16x^{-1}}{-1} \right]_1^2 + \frac{4}{3} \pi \\
&= \pi \left[ -\frac{16}{x} \right]_1^2 + \frac{4}{3} \pi \\
&= \left( -\frac{16}{2} + \frac{16}{1} \right) \pi + \frac{4}{3} \pi \\
&= 8\pi + \frac{4}{3} \pi \\
&= \frac{28}{3} \pi \text{ units}^3
\end{aligned}$$

### 9 Generated volume

= (Volume of the cone generated by the straight line from  $x = 0$  to  $x = 2$ ) - (Volume generated by the curve from  $x = 1$  to  $x = 2$ )

$$\begin{aligned}
&= \frac{1}{3} \pi r^2 h - \pi \int_1^2 y^2 dx \\
&= \frac{1}{3} \pi (1)^2 (2) - \pi \int_1^2 (x-1) dx \\
&= \frac{2}{3} \pi - \pi \left[ \frac{x^2}{2} - x \right]_1^2 \\
&= \frac{2}{3} \pi - \pi \left[ \frac{2^2}{2} - 2 - \left( \frac{1}{2} - 1 \right) \right] \\
&= \frac{2}{3} \pi - \frac{1}{2} \pi \\
&= \frac{1}{6} \pi \text{ units}^3
\end{aligned}$$

10 (a)  $y = 3x + 6 \dots (1)$

$y = 16 - x^2 \dots (2)$

$$3x + 6 = 16 - x^2$$

$$x^2 + 3x - 10 = 0$$

$$(x-2)(x+5) = 0$$

$$x = 2 \text{ or } x = -5$$

$x = -5$  is not accepted.

$$\therefore x = 2$$

At the turning point,  $x = 2$ ,  $y = 3(2) + 6 = 12$

Thus, the coordinates of point  $P$  are (2, 12).

(b) The straight line  $y = 3x + 6$  intersects the  $y$ -axis at (0, 6).

The curve  $y = 16 - x^2$  intersects the  $x$ -axis at (4, 0).

Area of the shaded region

= (Area under the straight line from  $x = 0$  to  $x = 2$ ) + (Area under the curve from  $x = 2$  to  $x = 4$ )

$$\begin{aligned}
&= \frac{1}{2} (6+12)(2) + \int_2^4 (16-x^2) dx \\
&= 18 + \left[ 16x - \frac{x^3}{3} \right]_2^4 \\
&= 18 + \left[ 16(4) - \frac{4^3}{3} - \left( 16(2) - \frac{2^3}{3} \right) \right] \\
&= 18 + \frac{128}{3} - \frac{88}{3} \\
&= \frac{94}{3} \text{ units}^2
\end{aligned}$$



(c) The y-intercept of the curve

$$y = 16 - x^2 \text{ is } 16.$$

Generated volume

$$\begin{aligned} &= \pi \int_{12}^{16} x^2 dy \\ &= \int_{12}^{16} \pi(16-y) dy \\ &= \pi \left[ 16y - \frac{y^2}{2} \right]_{12}^{16} \\ &= \pi \left[ 16(16) - \frac{16^2}{2} \right] - \pi \left[ 16(12) - \frac{12^2}{2} \right] \\ &= 128\pi - 120\pi \\ &= 8\pi \text{ units}^3 \end{aligned}$$

### UPSKILL 3.4

1 (a)  $\frac{dT}{dt} = \frac{1}{2}(14-t)$

At the turning point,  $T$  is a maximum,

$$\frac{dT}{dt} = 0$$

$$\frac{1}{2}(14-t) = 0$$

$$t = 14$$

$$\frac{d^2T}{dt^2} = -\frac{1}{2} \text{ (Negative)}$$

Thus,  $T$  is a maximum.

(b)  $\frac{dT}{dt} = \frac{1}{2}(14-t)$

$$T = \frac{1}{2} \int (14-t) dt$$

$$T = \frac{1}{2} \left( 14t - \frac{t^2}{2} \right) + c$$

Given that  $T = 25$  at the turning point where  $t = 8$ ,

$$25 = \frac{1}{2} \left( 14(8) - \frac{8^2}{2} \right) + c$$

$$25 = 40 + c$$

$$c = -15$$

$$\text{Thus, } T = \frac{1}{2} \left( 14t - \frac{t^2}{2} \right) - 15$$

At the turning point where  $t = 14$ , the maximum value of  $T$

$$= \frac{1}{2} \left[ 14(14) - \frac{14^2}{2} \right] - 15$$

$$= 34$$

2 (a) Sphere with a radius of 7 cm

(b) Volume of sphere

$$= \pi \int_{-7}^7 x^2 dy$$

$$= \pi \int_{-7}^7 (49 - y^2) dy$$

$$= \pi \left[ 49y - \frac{y^3}{3} \right]_{-7}^7$$

$$= \pi \left[ 49(7) - \frac{7^3}{3} - \left( (49)(-7) - \frac{(-7)^3}{3} \right) \right]$$

$$= \pi \left( \frac{686}{3} - \left( -\frac{686}{3} \right) \right)$$

$$= 457 \frac{1}{3} \pi \text{ cm}^3$$

### Summative Practice 3

1  $\int \left( t + \frac{1}{t} \right) \left( t - \frac{1}{t} \right) dt$

$$= \int \left( t^2 - \frac{1}{t^2} \right) dt$$

$$= \int (t^2 - t^{-2}) dt$$

$$= \frac{t^3}{3} - \frac{t^{-1}}{-1} + c$$

$$= \frac{t^3}{3} + \frac{1}{t} + c$$

2  $\int \frac{(1+x)(1-x)}{x^4} dx$

$$= \int \frac{1-x^2}{x^4} dx$$

$$= \int (x^{-4} - x^{-2}) dx$$

$$= \frac{x^{-3}}{-3} - \frac{x^{-1}}{-1} + c$$

$$= -\frac{1}{3x^3} + \frac{1}{x} + c$$

3  $\int \frac{10}{(5-2y)^3} dy$

$$= \int 10(5-2y)^{-3} dy$$

$$= 10 \left[ \frac{(5-2y)^{-2}}{-2(-2)} \right] + c$$

$$= \frac{5}{2(5-2y)^2} + c$$

$$\begin{aligned}
 4 \int \frac{3}{(1+2x)^4} dx &= \int 3(1+2x)^{-4} dx \\
 &= \left[ \frac{3(1+2x)^{-3}}{-3(2)} \right] + c \\
 &= -\frac{1}{2}(1+2x)^{-3} + c
 \end{aligned}$$

But the given integral is  $k(1+2x)^n + c$ .

By comparison,  $k = -\frac{1}{2}$  and  $n = -3$ .

$$\begin{aligned}
 5 \frac{d}{dx}[f(x)] &= 4g(x) \\
 f(x) &= 4 \int g(x) dx \\
 \int g(x) dx &= \frac{1}{4}[f(x)] + c
 \end{aligned}$$

$$\begin{aligned}
 6 \text{ (a)} \int_7^2 h(x) dx &= -\int_2^7 h(x) dx \\
 &= -4
 \end{aligned}$$

$$\begin{aligned}
 \text{(b)} \int_2^7 2h(x) dx - \int_2^7 p dx &= 6 \\
 2(4) - p[x]_2^7 &= 6 \\
 8 - p(7-2) &= 6 \\
 8 - 5p &= 6 \\
 -5p &= -2 \\
 p &= \frac{2}{5}
 \end{aligned}$$

$$\begin{aligned}
 7 \text{ (a)} \frac{dy}{dx} &= kx - 5 \\
 y &= \int (kx - 5) dx \\
 y &= \frac{kx^2}{2} - 5x + c \\
 \text{The curve passes through the point} & \\
 (1, -1), \text{ thus } x = 1 \text{ and } y = -1. & \\
 -1 &= \frac{k}{2} - 5 + c \\
 -2 &= k - 10 + 2c \\
 k + 2c &= 8 \dots (1)
 \end{aligned}$$

The curve passes through the point (0, 3), thus  $x = 0$  and  $y = 3$ .

$$\begin{aligned}
 y &= \frac{kx^2}{2} - 5x + c \\
 3 &= 0 - 0 + c \\
 c &= 3
 \end{aligned}$$

From (1) :  $k + 2(3) = 8 \Rightarrow k = 2$

$$\begin{aligned}
 \text{(b)} \frac{dy}{dx} &= 2x - 5 \\
 y &= \int (2x - 5) dx
 \end{aligned}$$

$$y = x^2 - 5x + h$$

The curve passes through the point (0, 3), thus  $x = 0$  and  $y = 3$ .

$$3 = 0 - 0 + h$$

$$h = 3$$

Thus,  $y = x^2 - 5x + 3$

$$\begin{aligned}
 8 \text{ (a)} \frac{dy}{dx} &= 2x + k \\
 y &= \int 2x + k \\
 y &= x^2 + kx + c
 \end{aligned}$$

The turning point is (1, 4).

$$\frac{dy}{dx} = 2(1) + k = 0$$

$$k = -2$$

(b) The curve passes through the point (1, 4), thus  $x = 1$  and  $y = 4$ .

$$y = x^2 - 2x + c$$

$$4 = 1^2 - 2(1) + c$$

$$c = 5$$

Thus, the curve intersects the y-axis at the point (0, 5).

$$\begin{aligned}
 9 \text{ Gradient of normal} &= \frac{1}{3-2x} \\
 \text{Gradient of tangent} &= -(3-2x)
 \end{aligned}$$

$$\frac{dy}{dx} = -3 + 2x$$

$$y = \int (-3 + 2x) dx$$

$$y = -3x + x^2 + c$$

The curve passes through the point (3, 4), thus  $x = 3$  and  $y = 4$ .

$$4 = -3(3) + 3^2 + c$$

$$c = 4$$

Hence, the equation of the curve is

$$y = -3x + x^2 + 4.$$

$$\begin{aligned}
 10 \text{ (a) Gradient of tangent} & \\
 &= \frac{2-0}{4-2} \\
 &= 1
 \end{aligned}$$

$$\text{(b)} \frac{dy}{dx} = 1$$

$$\frac{1}{2}x + p = 1$$

$$\frac{1}{2}(4) + p = 1$$

$$p = -1$$

(c)  $\frac{dy}{dx} = \frac{1}{2}x - 1$

$$y = \int \left( \frac{1}{2}x - 1 \right) dx$$

$$y = \frac{1}{2} \left( \frac{x^2}{2} \right) - x + c$$

$$y = \frac{1}{4}x^2 - x + c$$

The curve passes through the point (4, 2), thus  $x = 4$  and  $y = 2$ .

$$2 = \frac{1}{4}(4)^2 - 4 + c$$

$$2 = 4 - 4 + c$$

$$c = 2$$

Hence, the equation of the curve is

$$y = \frac{1}{4}x^2 - x + 2.$$

11 (a) The equation of the normal is

$$x + 2y - 5 = 0.$$

$$x + 2y - 5 = 0$$

$$2y = -x + 5$$

$$y = -\frac{1}{2}x + \frac{5}{2}$$

Since the gradient of the normal is

$-\frac{1}{2}$ , thus the gradient of the tangent is 2.

$$\frac{dy}{dx} = 2$$

$$\frac{k}{x^2} + 3 = 2$$

$$\frac{k}{1^2} + 3 = 2$$

$$k = -1$$

(b)  $\frac{dy}{dx} = -\frac{1}{x^2} + 3$

$$y = \int \left( -\frac{1}{x^2} + 3 \right) dx$$

$$y = \int \left( -x^{-2} + 3 \right) dx$$

$$y = \frac{-x^{-1}}{-1} + 3x + c$$

$$y = \frac{1}{x} + 3x + c$$

The curve passes through the point (1, 2), thus  $x = 1$  and  $y = 2$ .

$$y = \frac{1}{x} + 3x + c$$

$$2 = \frac{1}{1} + 3(1) + c$$

$$c = -2$$

Hence, the equation of the curve is

$$y = \frac{1}{x} + 3x - 2.$$

12 (a)  $\frac{dy}{dx} = 2x + 4$

$$y = \int (2x + 4) dx$$

$$y = x^2 + 4x + c$$

Given  $y = 7$  at the turning point,  $x = 1$ ,

$$7 = 1^2 + 4(1) + c$$

$$c = 2$$

The equation of the curve is

$$y = x^2 + 4x + 2.$$

(b)  $y = x^2 + 4x + 2$

$$\frac{dy}{dx} = 2x + 4$$

$$\frac{d^2y}{dx^2} = 2$$

$$x^2 \frac{d^2y}{dx^2} + (x-1) \frac{dy}{dx} + y + 3 = 0$$

$$x^2(2) + (x-1)(2x+4) + x^2 + 4x + 2 + 3 = 0$$

$$2x^2 + 2x^2 + 2x - 4 + x^2 + 4x + 5 = 0$$

$$5x^2 + 6x + 1 = 0$$

$$(5x+1)(x+1) = 0$$

$$x = -\frac{1}{5} \text{ or } -1$$

13 (a)  $\frac{dy}{dx} = 3x^2 - 4x$

$$y = \int (3x^2 - 4x) dx$$

$$y = x^3 - 2x^2 + c$$

The curve passes through the point (-1, -2), thus  $x = -1$  and  $y = -2$ .

$$-2 = (-1)^3 - 2(-1)^2 + c$$

$$-2 = -1 - 2 + c$$

$$c = 1$$

Hence, the equation of the curve is

$$y = x^3 - 2x^2 + 1.$$

(b)  $y = x^3 - 2x^2 + 1$

$$\frac{dy}{dx} = 3x^2 - 4x$$

$$\frac{d^2y}{dx^2} = 6x - 4$$

At the turning points,

$$\frac{dy}{dx} = 0$$

$$3x^2 - 4x = 0$$

$$x(3x - 4) = 0$$

$$x = 0 \text{ or } x = \frac{4}{3}$$

At the turning point,  $x = 0$ ,

$$y = 0^3 - 2(0)^2 + 1 = 1$$

Hence, (0, 1) is a turning point.

$$\frac{d^2y}{dx^2} = 6(0) - 4 = -2 \text{ (negative)}$$

Hence, (0, 1) is a maximum.

At the turning point,  $x = \frac{4}{3}$ .

$$y = \left(\frac{4}{3}\right)^3 - 2\left(\frac{4}{3}\right)^2 + 1 = -\frac{5}{27}$$

Hence,  $\left(\frac{4}{3}, -\frac{5}{27}\right)$  is a turning point.

$$\frac{d^2y}{dx^2} = 6\left(\frac{4}{3}\right) - 4 = 4 \text{ (positive)}$$

Hence,  $\left(\frac{4}{3}, -\frac{5}{27}\right)$  is a minimum point.

**14** (a)  $y = 6x - x^2$

At the turning point where  $x = 2$ ,

$$y = 6(2) - 2^2 = 8$$

$$\frac{dy}{dx} = 6 - 2x = 6 - 2(2) = 2$$

Hence, the equation of the tangent is

$$y - 8 = 2(x - 2)$$

$$y - 8 = 2x - 4$$

$$y = 2x + 4$$

(b) At the turning point,  $x = 0$ ,  $y = 4$

At the turning point,  $x = 6$ ,

$$y = 2(6) + 4 = 16$$

Area of the shaded region

$$= (\text{Area of the trapezium from } x = 0 \text{ to } x = 6) - (\text{Area under the curve from } x = 0 \text{ to } x = 6)$$

$$= \frac{1}{2}(4+16)(6) - \int_0^6 (6x - x^2) dx$$

$$= 60 - \left[ 3x^2 - \frac{x^3}{3} \right]_0^6$$

$$= 60 - \left[ 3(6)^2 - \frac{6^3}{3} \right]$$

$$= 60 - (108 - 72)$$

$$= 60 - 36$$

$$= 24 \text{ units}^2$$

**15** (a) At the  $x$ -axis,  $y = 0$

$$y = 27 - (x - 2)^3$$

$$0 = 27 - (x - 2)^3$$

$$(x - 2)^3 = 27$$

$$x - 2 = 3$$

$$x = 5$$

Thus,  $P$  is point (5, 0).

(b) Area of the shaded region

$$= \int_0^5 [27 - (x - 2)^3] dx$$

$$= \left[ 27x - \frac{(x - 2)^4}{4(1)} \right]_0^5$$

$$= \left[ 27(5) - \frac{3^4}{4} - \left( 27(0) - \frac{(-2)^4}{4} \right) \right]$$

$$= \frac{459}{4} - \left( -\frac{16}{4} \right)$$

$$= \frac{475}{4} \text{ units}^2$$

**16**  $y = 2x \quad \dots (1)$

$$y = 6x - x^2 \quad \dots (2)$$

Substitute (1) into (2) :

$$2x = 6x - x^2$$

$$x^2 - 6x + 2x = 0$$

$$x^2 - 4x = 0$$

$$x(x - 4) = 0$$

$$x = 0 \text{ or } 4$$

At the turning point,  $x = 0$ ,  $y = 0$

At the turning point,  $x = 4$ ,  $y = 2(4) = 8$

Thus, the points of intersection between the straight line  $y = 2x$  and the curve

$$y = 6x - x^2 \text{ are } (0, 0) \text{ and } (4, 8).$$

Area of the shaded region  $B$   
 = (Area of triangle from  $x = 0$  to  $x = 4$ )  
 + (Area under of curve from  $x = 4$  to  $x = 6$ )  
 =  $\left(\frac{1}{2} \times 4 \times 8\right) + \int_4^6 (6x - x^2) dx$   
 =  $16 + \left[3x^2 - \frac{x^3}{3}\right]_4^6$   
 =  $16 + \left[3(6)^2 - \frac{6^3}{3} - \left(3(4)^2 - \frac{4^3}{3}\right)\right]$   
 =  $16 + \left[36 - \frac{80}{3}\right]$   
 =  $16 + \frac{28}{3}$   
 =  $\frac{76}{3}$  units<sup>2</sup>

Area under of curve from  $x = 0$  to  $x = 6$   
 =  $\int_0^6 (6x - x^2) dx$   
 =  $\left[3x^2 - \frac{x^3}{3}\right]_0^6$   
 =  $3(6)^2 - \frac{6^3}{3} - 0$   
 =  $36$  units<sup>2</sup>

Area of the shaded region  $A$   
 =  $36 - \frac{76}{3}$   
 =  $\frac{32}{3}$  units<sup>2</sup>

Area of shaded region  $A$  : Area of shaded region  $B$   
 =  $\frac{\text{Area of region } A}{\text{Area of region } B}$   
 =  $\frac{\frac{32}{3}}{\frac{76}{3}}$   
 =  $\frac{8}{19}$   
 =  $8 : 19$  [Shown]

17 (a) Area of region  $A$   
 =  $\int_1^2 y dx$   
 =  $\int_1^2 \frac{8}{x^2} dx$

$$= \int_1^2 8x^{-2} dx$$

$$= \left[\frac{8x^{-1}}{-1}\right]_1^2$$

$$= \left[-\frac{8}{x}\right]_1^2$$

$$= -\frac{8}{2} - \left(-\frac{8}{1}\right)$$

$$= -4 + 8$$

$$= 4 \text{ units}^2$$

(b) Area of region  $B$  = Area of region  $C$

$$\int_2^p 8x^{-2} dx = \int_p^5 8x^{-2} dx$$

$$\left[-\frac{8}{x}\right]_2^p = \left[-\frac{8}{x}\right]_p^5$$

$$-\frac{8}{p} - \left(-\frac{8}{2}\right) = -\frac{8}{5} - \left(-\frac{8}{p}\right)$$

$$-\frac{8}{p} + 4 = -\frac{8}{5} + \frac{8}{p}$$

$$\frac{16}{p} = \frac{28}{5}$$

$$p = \frac{5}{28} \times 16$$

$$p = 2\frac{6}{7}$$

18 (a)  $y = (x-1)^2 = x^2 - 2x + 1$

$$\frac{dy}{dx} = 2x - 2$$

At the point  $P(3, 4)$ ,

$$m = \frac{dy}{dx} = 2(3) - 2 = 4$$

Equation of the tangent is

$$y - 4 = 4(x - 3)$$

$$y - 4 = 4x - 12$$

$$y = 4x - 8$$

At point  $Q$  ( $x$ -axis),

$$y = 4x - 8$$

$$0 = 4x - 8$$

$$4x = 8$$

$$x = 2$$

Thus,  $Q$  is point  $(2, 0)$ .

(b) Area of the shaded region  
 = (Area under the curve from  $x = 1$  to  $x = 3$ ) - (Area of triangle from  $x = 2$  to  $x = 3$ )

$$\begin{aligned}
&= \int_1^3 (x-1)^2 dx - \left(\frac{1}{2} \times 1 \times 4\right) \\
&= \left[ \frac{(x-1)^3}{3(1)} \right]_1^3 - 2 \\
&= \frac{2^3}{3} - 0 - 2 \\
&= \frac{2}{3} \text{ unit}^2
\end{aligned}$$

19 (a)  $y = x^3 - 3x + c$

$$\frac{dy}{dx} = 3x^2 - 3$$

At the minimum point  $P$ ,

$$\frac{dy}{dx} = 0$$

$$3x^2 - 3 = 0$$

$$3x^2 = 3$$

$$x^2 = 1$$

$$x = 1$$

Thus,  $P$  is point  $(1, 0)$ .

Substitute  $x = 1$  and  $y = 0$  into

$$y = x^3 - 3x + c$$

$$0 = 1^3 - 3(1) + c$$

$$c = 2$$

(b) Area of the shaded region

$$= \int_0^1 (x^3 - 3x + 2) dx$$

$$= \left[ \frac{x^4}{4} - \frac{3x^2}{2} + 2x \right]_0^1$$

$$= \frac{1}{4} - \frac{3}{2} + 2 - 0$$

$$= \frac{3}{4} \text{ unit}^2$$

20  $y = -x + 5 \quad \dots (1)$

$$y = x^2 - 2x + 5 \quad \dots (2)$$

Substitute (2) into (1) :

$$x^2 - 2x + 5 = -x + 5$$

$$x^2 - x = 0$$

$$x(x-1) = 0$$

$$x = 0 \text{ or } x = 1$$

The straight line  $y = -x + 5$  intersects the  $x$ -axis at  $(5, 0)$ .

At the turning point,  $x = 0$ ,  $y = -0 + 5 = 5$

At the turning point,  $x = 1$ ,  $y = -1 + 5 = 4$

Thus, the points of intersection between the curve and the straight line are  $(0, 5)$  and  $(1, 4)$ .

Area of the shaded region

$=$  (Area under the curve from  $x = 0$  to  $x = 1$ )  $+$  (Area of triangle from  $x = 1$  to  $x = 5$ )

$$= \int_0^1 (x^2 - 2x + 5) dx + \frac{1}{2} \times (5-1) \times 4$$

$$= \left[ \frac{x^3}{3} - x^2 + 5x \right]_0^1 + 8$$

$$= \left( \frac{1}{3} - 1 + 5 \right) + 8$$

$$= \frac{37}{3} \text{ units}^2$$

21  $x = 3 \quad \dots (1)$

$$y^2 = 4(x+1) \quad \dots (2)$$

Substitute (1) into (2) :

$$y^2 = 4(3+1) = 16$$

$$y = 4$$

Thus,  $B$  is point  $(3, 4)$  and  $A$  is point  $(0, 4)$ .

$$y^2 = 4(x+1)$$

At the  $y$ -axis,  $x = 0$ .

$$y^2 = 4(0+1)$$

$$y = 2$$

Thus, the curve intersects the  $y$ -axis at  $(0, 2)$ .

Area of the shaded region

$$= \int_2^4 x dy$$

$$= \int_2^4 \left( \frac{y^2}{4} - 1 \right) dy$$

$$= \left[ \frac{y^3}{12} - y \right]_2^4$$

$$= \frac{4^3}{12} - 4 - \left( \frac{2^3}{12} - 2 \right)$$

$$= \frac{4}{3} - \left( -\frac{4}{3} \right)$$

$$= \frac{8}{3} \text{ units}^2$$

$$\begin{aligned}
y^2 &= 4(x+1) \\
\frac{y^2}{4} &= x+1 \\
x &= \frac{y^2}{4} - 1
\end{aligned}$$

22 (a) Point  $(k, 3)$  lies on the straight line  $y = 5 - x$ .

$$\text{Thus, } 3 = 5 - k \Rightarrow k = 2.$$

(b) The straight line  $y = 5 - x$  intersects the  $y$ -axis at  $(0, 5)$ .

The curve  $y = (1 - x)(x - 5)$  intersects the  $x$ -axis at  $(1, 0)$  and  $(5, 0)$ .

Area of the shaded region  
 = (Area of trapezium from  $x = 0$  to  $x = 2$ ) - Area under the curve from  $x = 1$  to  $x = 2$ )

$$\begin{aligned} &= \frac{1}{2}(5+3)(2) - \int_1^2 y \, dx \\ &= 8 - \int_1^2 (1-x)(x-5) \, dx \\ &= 8 - \int_1^2 (-x^2 + 6x - 5) \, dx \\ &= 8 - \left[ -\frac{x^3}{3} + 3x^2 - 5x \right]_1^2 \\ &= 8 - \left[ -\frac{2^3}{3} + 3(2)^2 - 5(2) - \left( -\frac{1}{3} + 3 - 5 \right) \right] \\ &= 8 - \left[ -\frac{2}{3} - \left( -\frac{7}{3} \right) \right] \\ &= \frac{19}{3} \text{ units}^2 \end{aligned}$$

23 Area of the shaded region

$$\begin{aligned} &= \int_{-2}^2 [(-x^2 + 2x + 7) - (x^2 + 2x - 1)] \, dx \\ &= \int_{-2}^2 (-2x^2 + 8) \, dx \\ &= \left[ -\frac{2x^3}{3} + 8x \right]_{-2}^2 \\ &= \left[ -\frac{2(2)^3}{3} + 8(2) - \left( -\frac{2}{3}(-2)^3 + 8(-2) \right) \right] \\ &= \frac{32}{3} - \left( -\frac{32}{3} \right) \\ &= 21\frac{1}{3} \text{ units}^2 \end{aligned}$$

24  $y = x(x - 2) = x^2 - 2x$

$$y^2 = (x^2 - 2x)^2 = (x^4 - 4x^3 + 4x^2)$$

Generated volume

$$\begin{aligned} &= \pi \int_0^3 y^2 \, dx \\ &= \pi \int_0^3 (x^4 - 4x^3 + 4x^2) \, dx \\ &= \pi \left[ \frac{x^5}{5} - \frac{4x^4}{4} + \frac{4x^3}{3} \right]_0^3 \\ &= \pi \left[ \frac{x^5}{5} - x^4 + \frac{4x^3}{3} \right]_0^3 \\ &= \pi \left( \frac{3^5}{5} - 3^4 + 36 - 0 \right) \\ &= \pi \left( \frac{243}{5} - 81 + 36 \right) \\ &= \frac{18}{5} \pi \text{ units}^3 \end{aligned}$$

25 Generated volume

$$\begin{aligned} &= \pi \int_{-2}^1 x^2 \, dy \\ &= \pi \int_{-2}^1 (y^2 + 4)^2 \, dy \\ &= \pi \int_{-2}^1 (y^4 + 8y^2 + 16) \, dy \\ &= \pi \left[ \frac{y^5}{5} + \frac{8y^3}{3} + 16y \right]_{-2}^1 \\ &= \pi \left[ \frac{1}{5} + \frac{8}{3} + 16 - \left( -\frac{32}{5} + \frac{8}{3}(-8) - 32 \right) \right] \\ &= \pi \left[ \frac{283}{15} - \left( -\frac{896}{15} \right) \right] \\ &= \frac{393}{5} \pi \text{ units}^3 \end{aligned}$$

26 Generated volume

$$\begin{aligned} &= \pi \int_2^5 x^2 \, dy \\ &= \pi \int_2^5 (25 - y^2) \, dy \\ &= \pi \left[ 25y - \frac{y^3}{3} \right]_2^5 \\ &= \pi \left[ 25(5) - \frac{5^3}{3} - \left( 25(2) - \frac{2^3}{3} \right) \right] \\ &= \pi \left[ 125 - \frac{125}{3} - \left( 50 - \frac{8}{3} \right) \right] \end{aligned}$$

$$= \pi \left( \frac{250}{3} - \frac{142}{3} \right)$$

$$= 36\pi \text{ units}^3$$

27 (a) Generated volume

$$= \pi \int_0^4 x^2 dy$$

$$= \pi \int_0^4 (4-y) dy$$

$$= \pi \left[ 4y - \frac{y^2}{2} \right]_0^4$$

$$= \pi(16 - 8 - 0)$$

$$= 8\pi \text{ units}^3$$

(b) The curve intersects the  $x$ -axis at the points  $(-2, 0)$  and  $(2, 0)$ .

Generated volume

$$= 2\pi \int_0^2 y^2 dx$$

$$= 2\pi \int_0^2 (4-x^2)^2 dx$$

$$= 2\pi \int_0^2 (16-8x^2+x^4) dx$$

$$= 2\pi \left[ 16x - \frac{8x^3}{3} + \frac{x^5}{5} \right]_0^2$$

$$= 2\pi \left( 16(2) - \frac{8(2)^3}{3} + \frac{2^5}{5} - 0 \right)$$

$$= 2\pi \left( 32 - \frac{64}{3} + \frac{32}{5} - 0 \right)$$

$$= \frac{512}{15}\pi \text{ units}^3$$

28 (a) The curve intersects the  $x$ -axis at  $(2, 0)$  and the  $y$ -axis at the point  $(0, 5)$ .

Generated volume

$$= \pi \int_0^4 y^2 dx$$

$$= \pi \int_0^4 \frac{16-x^2}{4} dx$$

$$= \frac{1}{4}\pi \int_0^4 (16-x^2) dx$$

$$= \frac{1}{4}\pi \left[ 16x - \frac{x^3}{3} \right]_0^4$$

$$= \frac{1}{4}\pi \left[ 16(4) - \frac{4^3}{3} - 0 \right]$$

$$= \frac{1}{4}\pi \left( \frac{128}{3} \right)$$

$$= \frac{32}{3}\pi \text{ units}^2$$

(b) Generated volume

$$= \pi \int_0^4 x^2 dy$$

$$= \pi \int_0^2 (16-4y^2) dy$$

$$= \pi \left[ 16y - \frac{4y^3}{3} \right]_0^2$$

$$= \pi \left( 32 - \frac{32}{3} \right)$$

$$= \frac{64}{3}\pi \text{ units}^3$$

29  $y = 3\sqrt{x-1}$

$$y^2 = 9(x-1)$$

At the  $x$ -axis,  $y = 0$

$$0^2 = 9(x-1)$$

$$x = 1$$

Generated volume =  $\frac{9}{2}\pi \text{ units}^3$

$$\pi \int_1^k y^2 dx = \frac{9}{2}\pi$$

$$\int_1^k 9(x-1) dx = \frac{9}{2}$$

$$\int_1^k (x-1) dx = \frac{1}{2}$$

$$\left[ \frac{x^2}{2} - x \right]_1^k = \frac{1}{2}$$

$$\frac{k^2}{2} - k - \left( \frac{1}{2} - 1 \right) = \frac{1}{2}$$

$$\frac{k^2}{2} - k + \frac{1}{2} = \frac{1}{2}$$

$$\frac{k^2}{2} - k = 0$$

$$k^2 - 2k = 0$$

$$k(k-2) = 0$$

$$k = 2$$

30 Generated volume =  $2\pi$

$$\pi \int_{-3}^k x^2 dy = 2\pi$$

$$\int_{-3}^k (y+3) dy = 2$$



$$\left[\frac{y^2}{2} + 3y\right]_{-3}^k = 2$$

$$\frac{k^2}{2} + 3k - \left(\frac{9}{2} - 9\right) = 2$$

$$k^2 + 6k - (9 - 18) = 4$$

$$k^2 + 6k + 5 = 0$$

$$(k+1)(k+5) = 0$$

$$k = -5 \text{ or } k = -1$$

$k = -5$  is not accepted.  
 $\therefore k = -1$

$$= \pi + \frac{1}{7}\pi \left[16(4) - \frac{64}{3} - \left(48 - \frac{27}{3}\right)\right]$$

$$= \pi + \frac{1}{7}\pi \left[16(4) - \frac{64}{3} - \left(48 - \frac{27}{3}\right)\right]$$

$$= \pi + \frac{1}{7}\left(\frac{11}{3}\right)\pi$$

$$= \frac{32}{21}\pi \text{ units}^3$$

**31 (a)**  $3y = x \dots (1)$   
 $x^2 + 7y^2 = 16 \dots (2)$

Substitute (1) into (2) :

$$(3y)^2 + 7y^2 = 16$$

$$16y^2 = 16$$

$$y^2 = 1$$

$$y = \pm 1$$

$y = -1$  is not accepted.

$$\therefore y = 1$$

From (1) :

$$3(1) = x \Rightarrow x = 3$$

Thus,  $A$  is point  $(3, 1)$ .

$$x^2 + 7y^2 = 16$$

At the  $x$ -axis,  $y = 0$ .

$$x^2 + 7(0)^2 = 16$$

$$x = 4$$

Thus,  $B$  is point  $(4, 0)$ .

(b)  $x^2 + 7y^2 = 16$

$$y^2 = \frac{16 - x^2}{7}$$

Generated volume

= (Volume of cone generated by the straight line from  $x = 0$  to  $x = 3$ ) +  
 Volume generated by the curve from  $x = 3$  to  $x = 4$ )

$$= \frac{1}{3}\pi r^2 h + \pi \int_3^4 y^2 dx$$

$$= \frac{1}{3}\pi(1)(3) + \pi \int_3^4 \left(\frac{16 - x^2}{7}\right) dx$$

$$= \pi + \frac{1}{7}\pi \left[16x - \frac{x^3}{3}\right]_3^4$$