Form 5 Chapter 3
Integration
Fully-Worked Solutions

UPSKILL 3.2a

1 (a) $\int -10\,dx$
$= -10x + c$

(b) $\int 0\,dx$
$= 0x + c$
$= c$

(c) $\int dx$
$= 1x + c$
$= x + c$

(d) $\int 6x\,dx$
$= \frac{6x^2}{2} + c$
$= 3x^2 + c$

(e) $\int 6x^2\,dx$
$= 6\left(\frac{x^3}{3}\right) + c$
$= 2x^3 + c$

(f) $\int \frac{5}{3}x^4\,dx$
$= \frac{5}{3}\left(\frac{x^5}{5}\right) + c$
$= \frac{x^5}{3} + c$

(g) $\int \frac{6}{x^4}\,dx$
$= \int 6x^{-4}\,dx$
$= \frac{6x^{-3}}{-3} + c$
$= -2x^{-3} + c$
$= -\frac{2}{x^3} + c$

(h) $\int -\frac{2}{5x^3}\,dx$
$= \int -\frac{2}{5}x^{-3}\,dx$
$= -\frac{2}{5}\left(\frac{x^{-2}}{-2}\right) + c$
$= \frac{1}{5x^2} + c$

(i) $\int \frac{3}{x^7}\,dx$
$= \int 3x^{-7}\,dx$
$= 3\left(\frac{x^{-6}}{-6}\right) + c$
$= -\frac{1}{2x^6} + c$

(j) $\int \frac{5}{3x^6}\,dx$
$= \int \frac{5}{3}x^{-6}\,dx$
$= \frac{5}{3}\frac{x^{-5}}{(-5)} + c$
$= -\frac{1}{3x^5} + c$

2 (a) $\int (5x^4 + 3x^2 - 4)\,dx$
$= \frac{5x^5}{5} + \frac{3x^3}{3} - 4x + c$
$= x^5 + x^3 - 4x + c$

(b) $\int \left(\frac{2}{5}x^3 + \frac{1}{4x^3}\right)dx$
$= \int \left(\frac{2}{5}x^3 + \frac{1}{4}x^{-3}\right)dx$
$= \frac{2x^4}{20} + \frac{1}{4}\left(\frac{x^{-2}}{-2}\right) + c$
$= \frac{x^4}{10} - \frac{1}{8x^2} + c$

(c) $\int \left(3 - \frac{4}{x^2} + \frac{6}{x^3}\right) dx$

$= \int \left(3 - 4x^{-2} + 6x^{-3}\right) dx$

$= 3x - 4\frac{x^{-1}}{-1} + 6\left(\frac{x^{-2}}{-2}\right) + c$

$= 3x + \frac{4}{x} - \frac{3}{x^2} + c$

(d) $\int \left(2 + \frac{4}{x^2} - \frac{3}{x^4}\right) dx$

$= \int \left(2 + 4x^{-2} - 3x^{-4}\right) dx$

$= 2x + \frac{4x^{-1}}{-1} - \frac{3x^{-3}}{-3} + c$

$= 2x - \frac{4}{x} + \frac{1}{x^3} + c$

3 (a) $\int 2x^2(4x - 6)\, dx$

$= \int \left(8x^3 - 12x^2\right) dx$

$= \frac{8x^4}{4} - \frac{12x^3}{3} + c$

$= 2x^4 - 4x^3 + c$

(b) $\int (3x + 2)(2x - 1)\, dx$

$= \int \left(6x^2 + x - 2\right) dx$

$= \frac{6x^3}{3} + \frac{x^2}{2} - 2x + c$

$= 2x^3 + \frac{x^2}{2} - 2x + c$

(c) $\int (2 - 3x)^2\, dx$

$= \int \left(4 - 12x + 9x^2\right) dx$

$= 4x - \frac{12x^2}{2} + \frac{9x^3}{3} + c$

$= 4x - 6x^2 + 3x^3 + c$

(d) $\int \left(1 + \frac{2}{x}\right)\left(1 - \frac{2}{x}\right) dx$

$= \int \left(1 - \frac{4}{x^2}\right) dx$

$= \int \left(1 - 4x^{-2}\right) dx$

$= x - \frac{4x^{-1}}{-1} + c$

$= x + \frac{4}{x} + c$

4 (a) $\int \left(\frac{x^4 + 2}{x^2}\right)^2 dx$

$= \int \left(\frac{x^8 + 4x^4 + 4}{x^4}\right) dx$

$= \int \left(x^4 + 4 + 4x^{-4}\right) do$

$= \frac{x^5}{5} + 4x + \frac{4x^{-3}}{-3} + c$

$= \frac{x^5}{5} + 4x - \frac{4}{3x^3} + c$

(b) $\int \frac{(x + 4)(x - 4)}{x^2}\, dx$

$= \int \left(\frac{x^2 - 16}{x^2}\right) dx$

$= \int \left(1 - 16x^{-2}\right) dx$

$= x - \frac{16x^{-1}}{-1} + c$

$= x + \frac{16}{x} + c$

(c) $\int \frac{3x^3 - 3x^2 + 4}{x^2}\, dx$

$= \int \left(3x - 3 + 4x^{-2}\right) dx$

$= \frac{3x^2}{2} - 3x + \frac{4x^{-1}}{-1} + c$

$= \frac{3x^2}{2} - 3x - \frac{4}{x} + c$

UPSKILL 3.2b

1 (a) $\int (2x+1)^4\,dx$

$= \frac{(2x+1)^5}{5(2)} + c$

$= \frac{(2x+1)^5}{10} + c$

(b) $\int (2-x)^3\,dx$

$= \frac{(2-x)^4}{4(-1)} + c$

$= -\frac{(2-x)^4}{4} + c$

(c) $\int 4(3x-2)^5\,dx$

$= \frac{4(3x-2)^6}{6(3)} + c$

$= \frac{2}{9}(3x-2)^6 + c$

(d) $\int 3\left(\frac{1}{2}x-4\right)^3 dx$

$= 3\frac{\left(\frac{1}{2}x-4\right)^4}{4\left(\frac{1}{2}\right)} + c$

$= \frac{3}{2}\left(\frac{1}{2}x-4\right)^4 + c$

(e) $\int 2(3x-2)^{-2}\,dx$

$= \frac{2(3x-2)^{-1}}{-1(3)} + c$

$= -\frac{2}{3}(3x-2)^{-1} + c$

(f) $\int 2(4-3x)^{-6}\,dx$

$= \frac{2(4-3x)^{-5}}{-5(-3)} + c$

$= \frac{2}{15}(4-3x)^{-5} + c$

2 (a) $\int \frac{1}{(3x-4)^2}\,dx$

$= \int (3x-4)^{-2}\,dx$

$= \frac{(3x-4)^{-1}}{-1(3)} + c$

$= -\frac{1}{3(3x-4)} + c$

(b) $\int \frac{-2}{(2x-3)^4}\,dx$

$= \int -2(2x-3)^{-4}\,dx$

$= \frac{-2(2x-3)^{-3}}{-3(2)} + c$

$= \frac{1}{3(2x-3)^3} + c$

(c) $\int \frac{3}{(4x-3)^4}\,dx$

$= \int 3(4x-3)^{-4}\,dx$

$= \frac{3(4x-3)^{-3}}{-3(4)} + c$

$= -\frac{1}{4(4x-3)^3} + c$

(d) $\int \frac{5}{(4-3x)^3}\,dx$

$= \int 5(4-3x)^{-3}\,dx$

$= \frac{5(4-3x)^{-2}}{-2(-3)} + c$

$= \frac{5}{6(4-3x)^2} + c$

(e) $\int \frac{3}{4(6-x)^5}\,dx$

$= \int \frac{3(6-x)^{-5}}{4}\,dx$

$= \frac{3(6-x)^{-4}}{4(-4)(-1)}$

$= \frac{3}{16(6-x)^4} + c$

(f) $\int \frac{2}{3(4-2x)^2}\,dx$

$= \int \frac{2(4-2x)^{-2}}{3}\,dx$

$= \frac{2(4-2x)^{-1}}{3(-1)(-2)}$

$= \frac{1}{3(4-2x)} + c$

UPSKILL 3.2c

1 $\frac{dy}{dx} = x^2(4x-3)$

$y = \int \left(4x^3 - 3x^2\right) dx$

$y = \frac{4x^4}{4} - \frac{3x^3}{3} + c$

$y = x^4 - x^3 + c$

Since the curve passes through the point (2, 5), thus $x = 2$ and $y = 5$.

$5 = 2^4 - 2^3 + c$

$c = -3$

Hence, the equation of the curve is

$y = x^4 - x^3 - 3$.

2 $\frac{dy}{dx} = (2x-4)\,dx$

$y = \int (2x-4)\,dx$

$y = \frac{2x^2}{2} - 4x + c$

$y = x^2 - 4x + c$

Since the curve passes through the point (3, 0), thus $x = 3$ and $y = 0$.

$0 = 3^2 - 4(3) + c$

$c = 3$

Hence, the equation of the curve is

$y = x^2 - 4x + 3$.

3 $\frac{dy}{dx} = \frac{2}{x^2} + 1$

$y = \int \left(\frac{2}{x^2} + 1\right) dx$

$y = \int \left(2x^{-2} + 1\right) dx$

$y = \frac{2x^{-1}}{-1} + x + c$

$y = -\frac{2}{x} + x + c$

Since the curve passes through the point (1, 8), thus $x = 1$ and $y = 8$.

$8 = -\frac{2}{1} + 1 + c$

$c = 9$

Hence, the equation of the curve is

$y = -\frac{2}{x} + x + 9$.

4 $\frac{dy}{dx} = 3x^2(x+1)$

$y = \int 3x^2(x+1)\,dx$

$y = \int \left(3x^3 + 3x^2\right) dx$

$y = \frac{3x^4}{4} + \frac{3x^3}{3} + c$

$y = \frac{3x^4}{4} + x^3 + c$

Since the curve passes through the point (1, 2), thus $x = 1$ and $y = 2$.

$2 = \frac{3(1)^4}{4} + 1^3 + c$

$c = \frac{1}{4}$

Hence, the equation of the curve is

$y = \frac{3x^4}{4} + x^3 + \frac{1}{4}$.

5 $\frac{dy}{dx} = \frac{1}{(2x-1)^2}$

$y = \int \frac{1}{(2x-1)^2}\,dx$

$y = \int (2x-1)^{-2}\,dx$

$y = \frac{(2x-1)^{-1}}{-1(2)} + c$

$y = -\frac{1}{2(2x-1)} + c$

Since the curve passes through the point (2, 1), thus $x = 2$ and $y = 1$.

$1 = -\frac{1}{2[2(2)-1]} + c$

$1 = -\frac{1}{6} + c$

$c = \frac{7}{6}$

Hence, the equation of the curve is

$y = -\frac{1}{2(2x-1)} + \frac{7}{6}$.

6 $\frac{dy}{dx} = 2x - 4$

$y = \int (2x-4)\,dx$

$y = x^2 - 4x + c$

For minimum value,

$\frac{dy}{dx} = 0$

$2x - 4 = 0$

$x = 2$

The minimum value is 3.
This means that the value of $y = 3$.

$3 = 2^2 - 4(2) + c$

$c = 7$

Hence, the equation of the curve is
$y = x^2 - 4x + 7$.

UPSKILL 3.3a

1 (a) $\int_0^3 \left(2x^3 - 2x - 1\right) dx$

$= \left[\frac{2x^4}{4} - \frac{2x^2}{2} - x\right]_0^3$

$= \left[\frac{x^4}{2} - x^2 - x\right]_0^3$

$= \frac{3^4}{2} - 3^2 - 3 - 0$

$= \frac{57}{2}$

(b) $\int_{-1}^1 \left(3x^2 + x + \frac{1}{x^2}\right) dx$

$= \left[x^3 + \frac{x^2}{2} - \frac{1}{x}\right]_{-1}^1$

$= 1^3 + \frac{1^2}{2} - \frac{1}{1} - \left[(-1)^3 + \frac{(-1)^2}{2} - \frac{1}{-1}\right]$

$= \frac{1}{2} - \frac{1}{2}$

$= 0$

(c) $\int_0^2 (3x-1)\left(x^2+1\right) dx$

$= \int_0^2 \left(3x^3 + 3x - x^2 - 1\right) dx$

$= \left[\frac{3x^4}{4} + \frac{3x^2}{2} - \frac{x^3}{3} - x\right]_0^2$

$= \frac{3(2)^4}{4} + \frac{3(2)^2}{2} - \frac{2^3}{3} - 2 - 0$

$= 13\frac{1}{3}$

(d) $\int_1^2 \frac{3x^3+2}{x^3}\,dx$

$= \int_1^2 \left(3 + 2x^{-3}\right) dx$

$= \left[3x + \frac{2x^{-2}}{-2}\right]_1^2$

$= \left[3x - \frac{1}{x^2}\right]_1^2$

$= 3(2) - \frac{1}{2^2} - (3-1)$

$= \frac{15}{4}$

(e) $\int_0^2 x(x+3)(x-3)\,dx$

$= \int_0^2 x\left(x^2 - 9\right) dx$

$= \int_0^2 \left(x^3 - 9x\right) dx$

$= \left[\frac{x^4}{4} - \frac{9x^2}{2}\right]_0^2$

$= \frac{2^4}{4} - \frac{9(2)^2}{2} - 0$

$= -14$

(f) $\int_{-1}^{1} \left(2x - \frac{1}{x}\right)^2 \; dx$

$= \int_{-1}^{1} \left(4x^2 - 4 + x^{-2}\right) \; dx$

$= \left[\frac{4x^3}{3} - 4x + \frac{x^{-1}}{-1}\right]_{-1}^{1}$

$= \left[\frac{4x^3}{3} - 4x - \frac{1}{x}\right]_{-1}^{1}$

$= \frac{4}{3} - 4 - 1 - \left(-\frac{4}{3} + 4 + 1\right)$

$= -\frac{22}{3}$

(g) $\int_{0}^{2} (2x+1)^3 \; dx$

$= \left[\frac{(2x+1)^4}{4(2)}\right]_{0}^{2}$

$= \left[\frac{(2x+1)^4}{8}\right]_{0}^{2}$

$= \frac{5^4}{8} - \frac{1}{8}$

= 78

(h) $\int_{-1}^{1} \frac{1}{(3-x)^3} \; dx$

$\int_{-1}^{1} (3-x)^{-3} \; dx$

$= \left[\frac{(3-x)^{-2}}{-2(-1)}\right]_{-1}^{1}$

$= \left[\frac{1}{2(3-x)^2}\right]_{-1}^{1}$

$= \frac{1}{2(3-1)^2} - \frac{1}{2(3+1)^2}$

$= \frac{1}{8} - \frac{1}{32}$

$= \frac{3}{32}$

(i) $\int_{0}^{2} x(x-2)^2 \; dx$

$= \int_{0}^{2} x\left(x^2 - 4x + 4\right) \; dx$

$= \int_{0}^{2} \left(x^3 - 4x^2 + 4x\right) \; dx$

$= \left[\frac{x^4}{4} - \frac{4x^3}{3} + \frac{4x^2}{2}\right]_{0}^{2}$

$= \left[\frac{x^4}{4} - \frac{4x^3}{3} + 2x^2\right]_{0}^{2}$

$= \frac{2^4}{4} - \frac{4(2)^3}{3} + 2(2)^2 - 0$

$= \frac{4}{3}$

(j) $\int_{0}^{2} \frac{\left(3x^3 - 2x^5 + 2\right)}{x^3} \; dx$

$= \int_{0}^{2} \left(3 - 2x^2 + 2x^{-3}\right) \; dx$

$= \left[3x - \frac{2x^3}{3} + \frac{2x^{-2}}{-2}\right]_{1}^{2}$

$= \left[3x - \frac{2x^3}{3} - \frac{1}{x^2}\right]_{1}^{2}$

$= 3(2) - \frac{2}{3}(2)^3 - \frac{1}{2^2} - \left(3 - \frac{2}{3} - 1\right)$

$= \frac{5}{12} - \frac{4}{3}$

$= -\frac{11}{12}$

2 (a) $\int_{2}^{6} 5f(x) \; dx$

$= 5\int_{2}^{6} f(x) \; dx$

= 5(4)

= 20

(b) $\int_{2}^{6} [5 - f(x)] \; dx$

$= \int_{2}^{6} 5 \; dx - \int_{2}^{6} f(x) \; dx$

$= [5x]_{2}^{6} - 4$

$= 5(6-2) - 4$

= 16

(c) $\int_6^2 f(x)\ dx$

$= -\int_2^6 f(x)\ dx$

$= -4$

(d) $\int_2^3 f(x)\ dx + \int_3^4 f(x)\ dx + \int_4^6 f(x)\ dx$

$= \int_2^6 f(x)\ dx$

$= 4$

3 (a) $\int_2^4 g(x)\ dx - \int_7^4 g(x)\ dx$

$= \int_2^4 g(x)\ dx + \int_4^7 g(x)\ dx$

$= 6 + 8$

$= 14$

(b) $\int_2^7 [g(x) + 2x]\ dx$

$= \int_2^7 g(x)\ dx + \int_2^7 2x\ dx$

$= 6 + 8 + [x^2]_2^7$

$= 14 + 7^2 - 2^2$

$= 59$

(c) $\int_4^2 3\,g(x)\ dx + \int_4^7 2g(x)\ dx$

$= -3\int_2^4 g(x)\ dx + 2\int_4^7 g(x)\ dx$

$= -3(6) + 2(8)$

$= -2$

(d) $\int_2^4 [g(x) + kx]\,dx = 10$

$\int_2^4 g(x)\ dx + \int_2^4 kx\ dx = 10$

$6 + \frac{k}{2}[x^2]_2^4 = 10$

$\frac{k}{2}[4^2 - 2^2]_2^4 = 4$

$\frac{k}{2} \times 12 = 4$

$k = \frac{2}{3}$

4 $y = \frac{x}{x+4}$

$\frac{dy}{dx} = \frac{(x+4)(1) - x(1)}{(x+4)^2}$

$\frac{dy}{dx} = \frac{4}{(x+4)^2}$ [Shown]

$y = \int \frac{4}{(x+4)^2}\ dx$

$4[y]_2^3 = \int_2^3 \frac{16}{(x+4)^2}\ dx$

$4\left[\frac{x}{x+4}\right]_2^3 = \int_2^3 \frac{16}{(x+4)^2}\ dx$

$\int_2^3 \frac{16}{(x+4)^2}\ dx = 4\left[\frac{x}{x+4}\right]_2^3 =$

$= 4\left(\frac{3}{7} - \frac{2}{6}\right)$

$= \frac{8}{21}$

5 $y = \frac{x}{5-x}$

$\frac{dy}{dx} = \frac{(5-x)(1) - x(-1)}{(5-x)^2}$

$\frac{dy}{dx} = \frac{5}{(5-x)^2}$ [Shown]

$y = \int \frac{5}{(5-x)^2}\ dx$

$\frac{1}{10}[y]_1^3 = \frac{1}{10}\int_1^3 \frac{5}{(5-x)^2}\ dx$

$\int_1^3 \frac{1}{2(5-x)^2}\ dx = \frac{1}{10}\left[\frac{x}{5-x}\right]_1^3$

$= \frac{1}{10}\left(\frac{3}{2} - \frac{1}{4}\right)$

$= \frac{1}{8}$

6 $\frac{dy}{dx} = 4f(x)$

$y = 4\int f(x)\ dx$

$\frac{1}{4}y = \int f(x)\,dx$

$\int_1^2 f(x)\ dx = \frac{1}{4}[y]_1^2$

$= \frac{1}{4}\left[\frac{3x^2-1}{x}\right]_1^2$

$= \frac{1}{4}\left[\frac{3(2)^2-1}{2} - \frac{3(1)^2-1}{1}\right]$

$= \frac{1}{4}\left(\frac{11}{2} - 2\right)$

$= \frac{7}{8}$

7 $\frac{d}{dx}\left(\frac{x}{x^2-3}\right) = g(x)$

$\frac{x}{x^2-3} = \int g(x)\,dx$

$\left[\frac{x}{x^2-3}\right]_1^2 = \int_1^2 g(x)\ dx$

$\int_1^2 [2x - g(x)]\ dx = 2\left[\frac{x^2}{2}\right]_1^2 - \left[\frac{x}{x^2-3}\right]_1^2$

$= (2^2 - 1^2) - \left(\frac{2}{1} - \frac{1}{-2}\right)$

$= 3 - \frac{5}{2}$

$= \frac{1}{2}$

UPSKILL 3.3b

1 (a) Area of the shaded region

$= \int_{-1}^4 y\ dx$

$= \int_{-1}^4 (1+x)(4-x)\ dx$

$= \int_{-1}^4 (4 - x + 4x - x^2)\ dx$

$= \int_{-1}^4 (4 + 3x - x^2)\ dx$

$= \left[4x + \frac{3x^2}{2} - \frac{x^3}{3}\right]_{-1}^4$

$= \left[16 + \frac{3}{2}(16) - \frac{64}{3}\right] - \left[-4 + \frac{3}{2} - \left(-\frac{1}{3}\right)\right]$

$= \frac{56}{3} - \left(-\frac{13}{6}\right)$

$= 20\frac{5}{6}$ units^2

(b) Area of the shaded region

$= \int_1^3 y\ dx$

$= \int_1^3 \left(x + \frac{2}{x^2}\right)\ dx$

$= \int_1^3 (x + 2x^{-2})\ dx$

$= \left[\frac{x^2}{2} + 2\left(\frac{x^{-1}}{-1}\right)\right]_1^3$

$= \left[\frac{x^2}{2} - 2\left(\frac{1}{x}\right)\right]_1^3$

$= \frac{9}{2} - \frac{2}{3} - \left(\frac{1}{2} - 2\right)$

$= 5\frac{1}{3}$ units^2

(c) Area of the shaded region

$= \int_{-1}^2 y\ dx$

$= \int_{-1}^2 (x-1)^2\ dx$

$= \left[\frac{(x-1)^3}{3(1)}\right]_{-1}^2$

$= \left(\frac{1}{3} - \left(-\frac{8}{3}\right)\right)$

$= 3$ units^2

2 (a) Area of the shaded region

$= \left|\int_{-2}^2 y\ dx\right|$

$= \left|\int_{-2}^2 (x+2)(x-2)\ dx\right|$

$= \left|\int_{-2}^2 (x^2 - 4)\ dx\right|$

$$= \left| \left[\frac{x^3}{3} - 4x \right]_{-2}^{2} \right|$$

$$= \left| \frac{8}{3} - 8 - \left(\frac{-8}{3} + 8 \right) \right|$$

$$= \left| -10\frac{2}{3} \right|$$

$$= 10\frac{2}{3} \text{ units}^2$$

(b) Area of the shaded region

$$= \left| \int_0^3 y \, dx \right|$$

$$= \left| \int_0^3 (x-3)^3 \, dx \right|$$

$$= \left| \left[\frac{(x-3)^4}{4(1)} \right]_0^3 \right|$$

$$= \left| 0 - \left(\frac{81}{4} \right) \right|$$

$$= \frac{81}{4} \text{ units}^2$$

3 (a) $y = x(x+2)(x-1)$

$$= x\left(x^2 + x - 2\right)$$

$$= x^3 + x^2 - 2x$$

Area of the shaded region

$$= \int_{-2}^{0} \left(x^3 + x^2 - 2x\right) dx + \left| \int_0^1 \left(x^3 + x^2 - 2x\right) dx \right|$$

$$= \left[\frac{x^4}{4} + \frac{x^3}{3} - x^2 \right]_{-2}^{0} + \left| \left[\frac{x^4}{4} + \frac{x^3}{3} - x^2 \right]_0^1 \right|$$

$$= 0 - \left(\frac{16}{4} - \frac{8}{3} - 4 \right) + \left| \left(\frac{1}{4} + \frac{1}{3} - 1 - 0 \right) \right|$$

$$= \frac{8}{3} + \left| -\frac{5}{12} \right|$$

$$= 3\frac{1}{12} \text{ units}^2$$

(b) $y = x(1+x)(4-x)$

$$y = x\left(4 + 3x - x^2\right)$$

$$y = 4x + 3x^2 - x^3$$

Area of the shaded region

$$= \left| \int_{-1}^{0} \left(4x + 3x^2 - x^3\right) dx \right| + \int_0^4 \left(4x + 3x^2 - x^3\right) dx$$

$$= \left| \left[2x^2 + x^3 - \frac{x^4}{4} \right]_{-1}^{0} \right| + \left[2x^2 + x^3 - \frac{x^4}{4} \right]_0^4$$

$$= \left| 0 - \left(2 - 1 - \frac{1}{4} \right) \right| + 2(16) + 64 - \frac{4^4}{4} - 0$$

$$= \left| -\frac{3}{4} \right| + 32$$

$$= 32\frac{3}{4} \text{ units}^2$$

(c) Area of the shaded region

$$= \int_0^3 \left(3 + 2x - x^2\right) dx + \left| \int_3^4 \left(3 + 2x - x^2\right) dx \right|$$

$$= \left[3x + x^2 - \frac{x^3}{3} \right]_0^3 + \left| \left[3x + x^2 - \frac{x^3}{3} \right]_3^4 \right|$$

$$= 9 + 9 - 9 + \left| 3(4) + 4^2 - \frac{64}{3} - \left[3(3) + 3^2 - \frac{27}{3} \right] \right|$$

$$= 9 + \left| \frac{20}{3} - 9 \right|$$

$$= 9 + \frac{7}{3}$$

$$= 11\frac{1}{3} \text{ units}^2$$

4 (a) Area of the shaded region

$$= \int_{-1}^{3} x \, dy$$

$$= \int_{-1}^{3} \left(3y^2 + 2\right) dy$$

$$= \left[\frac{3y^3}{3} + 2y \right]_{-1}^{3}$$

$$= \left[y^3 + 2y \right]_{-1}^{3}$$

$$= 27 + 6 - (-1 - 2)$$

$$= 36 \text{ units}^2$$

(b) Area of the shaded region

$= \int_0^3 x\, dy$

$= \int_0^3 y^2(3 - y)\, dy$

$= \int_0^3 \left(3y^2 - y^3\right) dy$

$= \left[y^3 - \frac{y^4}{4} \right]_0^3$

$= 3^3 - \frac{3^4}{4}$

$= 6\frac{3}{4}$ units^2

5 (a) Area of the shaded region

$= \left| \int_{-1}^3 x\, dy \right|$

$= \left| \int_{-1}^3 \left(y^2 - 2y - 3\right) dy \right|$

$= \left| \left[\frac{y^3}{3} - y^2 - 3y \right]_{-1}^3 \right|$

$= \left| 9 - 9 - 9 - \left(-\frac{1}{3} - 1 + 3 \right) \right|$

$= \left| -9 - \frac{5}{3} \right|$

$= \left| -10\frac{2}{3} \right|$

$= 10\frac{2}{3}$ units^2

(b) $x = y(y+1)(y-1)$

$x = y\left(y^2 - 1\right)$

$x = \left(y^3 - y\right)$

Area of the shaded region

$= \int_{-1}^0 \left(y^3 - y\right) dy + \left| \int_0^1 \left(y^3 - y\right) dy \right|$

$= \left[\frac{y^4}{4} - \frac{y^2}{2} \right]_{-1}^0 + \left| \left[\frac{y^4}{4} - \frac{y^2}{2} \right]_0^1 \right|$

$= 0 - \left(\frac{1}{4} - \frac{1}{2} \right) + \left| \frac{1}{4} - \frac{1}{2} - 0 \right|$

$= \frac{1}{4} + \frac{1}{4}$

$= \frac{1}{2}$ unit^2

6 (a) (i) $y = -x + 4$ … (1)

$y = 4 + 3x - x^2$ … (2)

Substitute (1) into (2) :

$-x + 4 = 4 + 3x - x^2$

$x^2 - 4x = 0$

$x(x - 4) = 0$

$x = 0$ or $x = 4$

At the turning point, $x = 0$,
$y = -0 + 4 = 4$

At the turning point, $x = 4$,
$y = -4 + 4 = 0$

Thus, the coordinates of the points P and Q are (0, 4) and (4, 0) respectively.

(ii) Area of the shaded region

= (Area under the curve from $x = 0$ to $x = 4$) – (Area of the triangle from $x = 0$ to $x = 4$)

$= \int_0^4 (4 + 3x - x^2)\, dx - \frac{1}{2} \times 4 \times 4$

$= \left[4x + \frac{3x^2}{2} - \frac{x^3}{3} \right]_0^4 - 8$

$= 4(4) + \frac{3}{2}(4)^2 - \frac{4^3}{3} - 8$

$= \frac{56}{3} - 8$

$= 10\frac{2}{3}$ units^2

(b) (i) $y = 21$ ... (1)

$y = 10x - x^2$ … (2)

$21 = 10x - x^2$

$x^2 - 10x + 21 = 0$

$(x - 3)(x - 7) = 0$

$x = 3$ or $x = 7$

Thus, the coordinates of the points P and Q are (3, 21) and (7, 21) respectively.

(ii) Area of the shaded region

= (Area under the curve from $x = 3$ to $x = 7$) – (Area of the rectangle from $x = 3$ to $x = 7$)

$= \int_3^7 \left(10x - x^2\right) dx - (4 \times 21)$

$= \left[5x^2 - \frac{x^3}{3} \right]_3^7 - 84$

$= \left[5(7)^2 - \frac{7^3}{3} - \left(5(3)^2 - \frac{3^3}{3} \right) \right] - 84$

$=\left(\frac{392}{3}-36\right)-84$

$= 10\frac{2}{3}$ units^2

(c) (i) $x=9$ … (1)

$y^2=4x$ … (2)

Substitute (1) into (2) :

$y^2=4(9) = 36$

$y=\pm 6$

Thus, the coordinates of the points P and Q are (9, 6) and (9, –6) respectively.

(ii) Area of the shaded region

= (Area of the rectangle from $y = -6$ to $y = 6$) – (Area between the curve and the y-axis from $y = -6$ to $y = 6$)

$=12\times 9 - \int_{-6}^{6}\frac{y^2}{4}\,dy$

$= 108 - \left[\frac{y^3}{12}\right]_{-6}^{6}$

$= 108 - \left[\frac{6^3}{12}-\left(\frac{-6^3}{12}\right)\right]$

= 108 – (18 + 18)

= 72 units^2

(d) (i) $y=4x+1$ … (1)

$y=x^2+5$ … (2)

Substitute (2) into (1) :

$x^2+5=4x+1$

$x^2-4x+4=0$

$(x-2)(x-2)=0$

$x = 2$

From (1) : $y=4(2)+1=9$

Thus, the coordinates of point Q are (2, 9).

$y=4x+1$

At the y-axis, $x = 0$.

$y = 4(0) + 1 = 1$

Thus, the coordinates of point P are (0, 1).

(ii) Area of the shaded region

= (Area under the curve from $x = 0$ to $x = 2$) – (Area of trapezium from $x = 0$ to $x = 2$)

$= \int_{0}^{2}\left(x^2+5\right)-\frac{1}{2}(1+9)(2)$

$= \left[\frac{x^3}{3}+5x\right]_0^2 -10$

$=\frac{8}{3}+10-10$

$=\frac{8}{3}$ units^2

7 (a) (i) $y=x+3$ … (1)

$y=2x^2$ … (2)

Substitute (2) into (1) :

$2x^2=x+3$

$2x^2-x-3=0$

$x=\frac{3}{2}$ or $x=-1$

$x=\frac{3}{2}$ is not accepted.

$\therefore x=-1$

From (1) : $y=-1+3=2$

Thus, the coordinates of point Q are $(-1, 2)$.

$y=x+3$

At the x-axis, $y = 0$.

$y=x+3$

$0 = x + 3$

$x=-3$

Thus, the coordinates of point P are (–3, 0).

(ii) Area of the shaded region

= (Area of the triangle from $x = -3$ to $x = -1$) + (Area under the curve from $x = -1$ to $x = 0$)

$= \frac{1}{2}\times 2\times 2+\int_{-1}^{0} 2x^2\,dx$

$=2+\left[\frac{2x^3}{3}\right]_{-1}^{0}$

$=2+\frac{2}{3}\left[0-(-1)\right]$

$=2+\frac{2}{3}$

$=2\frac{2}{3}$ units^2

(b) (i) $y=-x+6$ (1)

$2x=(y-2)^2$ ` (2)

Substitute (1) into (2) :

$2x=(-x+6-2)^2$

$2x = (-x+4)^2$
$2x = x^2 - 8x + 16$
$x^2 - 10x + 16 = 0$
$(x-2)(x-8) = 0$
$x = 2$ or $x = 8$
$x = 8$ is not accepted.
$\therefore x = 2$

From (1) : $y = -2 + 6 = 4$
Thus, the coordinates of point Q are $(2, 4)$.

$x + y = 6$
At the y-axis, $x = 0$.
$0 + y = 6$
$y = 6$
Thus, the coordinates of point P are $(0, 6)$.

(ii) Area of the shaded region
= (Area between the curve and the y-axis from $y = 2$ to $y = 4$) + (Area of the triangle from $y = 4$ to $y = 6$)
$= \int_2^4 \frac{(y-2)^2}{2}\, dy + \frac{1}{2} \times 2 \times 2$
$= \left[\frac{(y-2)^3}{2(3)(1)}\right]_2^4 + 2$
$= \frac{1}{6}\left(2^3 - 0\right) + 2$
$= \frac{10}{3}$ units^2

8 Area of the shaded region = 81
$\int_0^k y\, dx = 81$
$\int_0^k 4x^3\, dx = 81$
$\left[x^4\right]_0^k = 81$
$k^4 - 0 = 81$
$k = 3$

9 Area of the shaded region = $\frac{1}{3}$
$\int_0^k x\, dy = \frac{1}{3}$
$\int_0^k \frac{y^2}{8}\, dx = \frac{1}{3}$
$\left[\frac{y^3}{24}\right]_0^k = \frac{1}{3}$
$k^3 = \frac{24}{3}$
$k^3 = 8$
$k = 2$

10 Area of the shaded region
$= \int_0^3 \left[\left(-2x^2 + 6x\right) - \left(x^2 - 3x\right)\right] dx$
$= \int_0^3 \left(-3x^2 + 9x\right) dx$
$= \left[-x^3 + \frac{9x^2}{2}\right]_0^3$
$= \left(-3^3 + \frac{9}{2} \times 3^2 - 0\right)$
$= 13\frac{1}{2}$ units^2

11 Area of the shaded region
$= \int_{-1}^1 \left[\left(x^3 - x\right) - \left(x^2 - 1\right)\right] dx$
$= \int_{-1}^1 \left(x^3 - x^2 - x + 1\right) dx$
$= \left[\frac{x^4}{4} - \frac{x^3}{3} - x^2 + x\right]_{-1}^1$
$= \frac{1}{4} - \frac{1}{3} - 1 + 1 - \left(\frac{1}{4} + \frac{1}{3} - 1 - 1\right)$
$= -\frac{1}{12} - \left(-\frac{17}{12}\right)$
$= \frac{4}{3}$ units^2

UPSKILL 3.3c

1 (a) Generated volume
$= \pi \int_1^4 y^2\, dx$
$= \pi \int_1^4 \left(\frac{4}{x}\right)^2 dx$
$= \pi \int_1^4 \frac{16}{x^2}\, dx$
$= \pi \int_1^4 16x^{-2}\, dx$
$= \pi \left[\frac{16x^{-1}}{-1}\right]_1^4$
$= \pi \left[-\frac{16}{x}\right]_1^4$

$$= \pi\left[-\frac{16}{4}-(-16)\right]$$
$$= 12\pi \text{ units}^3$$

(b) Generated volume
$$= \pi\int_1^3 -\left(x+\frac{2}{x}\right)^2 dx$$
$$= \pi\int_1^3 \left(x^2+4+4x^{-2}\right) dx$$
$$= \pi\left[\frac{x^3}{3}+4x+\frac{4x^{-1}}{-1}\right]_1^3$$
$$= \pi\left[\frac{x^3}{3}+4x-\frac{4}{x}\right]_1^3$$
$$= \pi\left[9+12-\frac{4}{3}-\left(\frac{1}{3}+4-4\right)\right]$$
$$= \pi\left(\frac{59}{3}-\frac{1}{3}\right)$$
$$= \frac{58}{3}\pi \text{ units}^2$$

(c) Generated volume
$$= \pi\int_{-1}^2 \left(4-x^2\right)^2$$
$$= \pi\int_{-1}^2 \left(16-8x^2+x^4\right) dx$$
$$= \pi\left[16x-\frac{8x^3}{3}+\frac{x^5}{5}\right]_{-1}^2$$
$$= \pi\left[32-\frac{64}{3}+\frac{32}{5}-\left(-16+\frac{8}{3}-\frac{1}{5}\right)\right]$$
$$= \frac{153}{5}\pi \text{ units}^2$$

(d) Generated volume
$$= \pi\int_0^2 \left(x^2-3x\right)^2 dx$$
$$= \pi\int_0^2 \left(x^4-6x^3+9x^2\right) dx$$
$$= \pi\left[\frac{x^5}{5}-\frac{6x^4}{4}+\frac{9x^3}{3}\right]_0^2$$
$$= \pi\left[\frac{x^5}{5}-\frac{3x^4}{2}+3x^3\right]_0^2$$
$$= \pi\left[\frac{32}{5}-\frac{3(16)}{2}+3(8)-0\right]$$
$$= \frac{32}{5} \text{ units}^2$$

(e) Generated volume
$$= \pi\int_0^3 \left[(x-2)^2\right]^2 dx$$
$$= \pi\left[\frac{(x-2)^5}{5(1)}\right]_0^3$$
$$= \pi\left[\frac{1}{5}-\left(-\frac{32}{5}\right)\right]$$
$$= \frac{33}{5}\pi \text{ units}^2$$

(f) Generated volume
$$= \pi\int_{\frac{1}{2}}^1 y^2 dx$$
$$= \pi\int_{\frac{1}{2}}^1 \left(4x-\frac{1}{x}\right)^2 dx$$
$$= \pi\int_{\frac{1}{2}}^1 \left(16x^2-8+x^{-2}\right) dx$$
$$= \pi\left[\frac{16x^3}{3}-8x+\frac{x^{-1}}{-1}\right]_{\frac{1}{2}}^1$$
$$= \pi\left[\frac{16x^3}{3}-8x-\frac{1}{x}\right]_{\frac{1}{2}}^1$$
$$= \pi\left[\frac{16}{3}-8-1-\left[\left(\frac{16}{3}\right)\left(\frac{1}{8}\right)-\frac{8}{2}-2\right]\right]$$
$$= \pi\left[-\frac{11}{3}-\left(-\frac{16}{3}\right)\right]$$
$$= \frac{5}{3}\pi \text{ units}^2$$

2 (a) Generated volume
$$= \pi\int_0^2 x^2 dy$$
$$= \pi\int_0^2 \left(\frac{y^2}{4}\right)^2 dy$$
$$= \pi\int_0^2 \frac{y^4}{16} dy$$
$$= \pi\left[\frac{y^5}{16(5)}\right]_0^2$$
$$= \pi\left(\frac{32}{80}-0\right)$$
$$= \frac{2}{5}\pi \text{ units}^3$$

(b) Generated volume

$= \pi \int_0^1 x^2 \, dy$

$= \pi \int_0^1 \left[(y-1)^2\right]^2 dy$

$= \pi \int_0^1 (y-1)^4 \, dy$

$= \pi \left[\frac{(y-1)^5}{5(1)}\right]_0^1$

$= \pi \left[0 - \left(-\frac{1}{5}\right)\right]$

$= \frac{1}{5}\pi$ units^3

(c) Generated volume

$= \pi \int_0^1 x^2 \, dy$

$= \pi \int_0^1 \left(y^2 - 2y\right)^2 \, dy$

$= \pi \int_0^2 \left(y^4 - 4y^3 + 4y^2\right) dy$

$= \pi \left[\frac{y^5}{5} - y^4 + \frac{4y^3}{3}\right]_0^2$

$= \frac{32}{5} - 16 + \frac{32}{3} - 0$

$= \frac{16}{15}\pi$ units^3

(d) Generated volume

$= \pi \int_1^2 x^2 \, dy$

$= \pi \int_1^2 \left(\frac{y^2}{6}\right)^2 dy$

$= \pi \int_1^2 \left(\frac{y^4}{36}\right) dy$

$= \pi \left[\frac{y^5}{180}\right]_1^2$

$= \pi \left[\frac{2^5}{180} - \frac{1}{180}\right]_1^2$

$= \frac{31}{180}\pi$ units^3

(e) Generated volume

$= \pi \int_0^9 (9 - y) \, dy$

$= \pi \left[9y - \frac{y^2}{2}\right]_0^9$

$= \pi \left(81 - \frac{81}{2}\right)$

$= \frac{81}{2}\pi$ units^3

(f) Generated volume

$= \pi \int_0^3 x^2 \, dy$

$= \pi \int_0^3 \left[-(y-3)^2\right]^2 dy$

$= \pi \int_0^3 \left[(y-3)^4\right] dy$

$= \pi \left[\frac{(y-3)^5}{5(1)}\right]_0^3$

$= \pi \left[0 - \left(-\frac{243}{5}\right)\right]$

$= \frac{243}{5}\pi$ units^3

3 Generated volume = π

$\pi \int_1^k y^2 \, dx = \pi$

$\int_1^k \left(\frac{-\sqrt{2x+1}}{2}\right)^2 dx = 1$

$\int_1^k \left(\frac{2x+1}{4}\right) dx = 1$

$\int_1^k (2x+1) \, dx = 4$

$\left[x^2 + x\right]_1^k = 4$

$k^2 + k - (1+1) = 4$

$k^2 + k - 6 = 0$

$(k+3)(k-2) = 0$

$k = -2$ or $k = 2$

$k = -2$ is not accepted because k has to be positive.

$\therefore k = 2$

4 Generated volume = $\frac{9}{2}\pi$

$$\pi\int_k^4 x^2\,dy = \frac{9}{2}\pi$$

$$\int_k^4 (4-y)\,dy = \frac{9}{2}$$

$$\left[4y-\frac{y^2}{2}\right]_k^4 = \frac{9}{2}$$

$$4(4)-\frac{4^2}{2}-\left(4k-\frac{k^2}{2}\right)=\frac{9}{2}$$

$$8-4k+\frac{k^2}{2}-\frac{9}{2}=0$$

$$16-8k+k^2-9=0$$

$$k^2-8k+7=0$$

$$(k-7)(k-1)=0$$

$$k=7 \text{ or } k=1$$

$k=7$ is not accepted because $k < 4$.

$\therefore k=1$

5 Generated volume = 2π

$$\pi\int_2^k x^2\,dy = 2\pi$$

$$\int_2^k (y-2)\,dy = 2$$

$$\left[\frac{y^2}{2}-2y\right]_2^k = 2$$

$$\frac{k^2}{2}-2k-(2-4)=2$$

$$\frac{k^2}{2}-2k+2=2$$

$$k^2-4k=0$$

$$k(k-4)=0$$

$$k=0 \text{ or } k=4$$

$k=0$ is not accepted.

$\therefore k=4$

6 $y=\frac{8}{x}$

At the turning point, $y=\frac{4}{3}$,

$$\frac{4}{3}=\frac{8}{x}$$

$$4x=24$$

$$x=6$$

Generated volume

= (Volume generated by the curve from x = 2 to x = 6) – (Volume generated by $y=\frac{4}{3}$ from x = 2 to x = 6)

= $\pi\int_2^6 \left(\frac{8}{x}\right)^2 dx$ – (Volume of cylinder generated with a radius of $\frac{4}{3}$ units and a height of 4 units)

$$= \pi\int_2^6 64x^{-2}\,dx - \pi r^2 h$$

$$=\pi\left[\frac{64x^{-1}}{-1}\right]_2^6 - \pi\left(\frac{4}{3}\right)^2(4)$$

$$=\pi\left[-\frac{64}{x}\right]_2^6 + \frac{64}{9}\pi$$

$$=\pi\left[-\frac{64}{6}-\left(-\frac{64}{2}\right)\right]-\frac{64}{9}\pi$$

$$=\frac{64}{3}\pi-\frac{64}{9}\pi$$

$$=\frac{128}{9}\pi \text{ units}^3$$

7 (a) $y=6x-x^2$

$$\frac{dy}{dx}=6-2x$$

At maximum point,

$$\frac{dy}{dx}=0$$

$$6-2x=0$$

$$x=3$$

At the turning point $x=3$,

$$y=6(3)-3^2=9$$

Thus, the coordinates of the maximum point are $Q(3, 9)$.

(b) Generated volume

= (Volume of the cylinder generated by the straight line y = 9 from x = 0 to x = 3) – (Volume generated by the curve from x = 0 to x = 3)

$$= \pi r^2 h - \pi\int_0^3 \left(6x-x^2\right)^2 dx$$

$$=\pi(9)^2(3)-\pi\int_0^3 \left(36x^2-12x^3+x^4\right) dx$$

$$=243\pi-\pi\left[\frac{36x^3}{3}-\frac{12x^4}{4}+\frac{x^5}{5}\right]_0^3$$

$$= 243\pi - \pi\left[\frac{36x^3}{3} - \frac{12x^4}{4} + \frac{x^5}{5}\right]_0^3$$

$$= 243\pi - \pi\left(12(3)^3 - 3(3)^4 + \frac{3^5}{5} - 0\right)$$

$$= 243\pi - \frac{648}{5}\pi$$

$$= \frac{567}{5}\pi \text{ units}^3$$

8 $y = \frac{4}{x}$ … (1)

$y = -2x + 6$ … (2)

Substitute (1) into (2) :

$\frac{4}{x} = -2x + 6$

$4 = -2x^2 + 6x$

$2x^2 - 6x + 4 = 0$

$x^2 - 3x + 2 = 0$

$(x-1)(x-2) = 0$

$x = 1$ or $x = 2$

At the turning point, $x = 1$, $y = \frac{4}{1} = 4$

At the turning point, $x = 2$, $y = \frac{4}{2} = 2$

Thus, the points of intersection between the straight line and the curve are (1, 4) and (2, 2).

The straight line $y = -2x + 6$ intersects the x-axis at (3, 0).

(Volume generated by the curve from x = 1 to x = 2) + (Volume of the cone generated by the straight line from x = 2 to x = 3)

$$= \pi\int_1^2 \left(\frac{4}{x}\right)^2 dx + \frac{1}{3}\pi r^2 h$$

$$= \pi\int_1^2 \left(16x^{-2}\right) dx + \frac{1}{3}\pi(2)^2(3-2)$$

$$= \pi\left[\frac{16x^{-1}}{-1}\right]_1^2 + \frac{4}{3}\pi$$

$$= \pi\left[-\frac{16}{x}\right]_1^2 + \frac{4}{3}\pi$$

$$= \left(-\frac{16}{2} + \frac{16}{1}\right)\pi + \frac{4}{3}\pi$$

$$= 8\pi + \frac{4}{3}\pi$$

$$= \frac{28}{3}\pi \text{ units}^3$$

9 Generated volume

= (Volume of the cone generated by the straight line from x = 0 to x = 2) – (Volume generated by the curve from x = 1 to x = 2)

$$= \frac{1}{3}\pi r^2 h - \pi\int_1^2 y^2\, dx$$

$$= \frac{1}{3}\pi(1)^2(2) - \pi\int_1^2 (x-1)\, dx$$

$$= \frac{2}{3}\pi - \pi\left[\frac{x^2}{2} - x\right]_1^2$$

$$= \frac{2}{3}\pi - \pi\left[\frac{2^2}{2} - 2 - \left(\frac{1}{2} - 1\right)\right]$$

$$= \frac{2}{3}\pi - \frac{1}{2}\pi$$

$$= \frac{1}{6}\pi \text{ units}^3$$

10 (a) $y = 3x + 6$ … (1)

$y = 16 - x^2$ … (2)

$3x + 6 = 16 - x^2$

$x^2 + 3x - 10 = 0$

$(x-2)(x+5) = 0$

$x = 2$ or $x = -5$

$x = -5$ is not accepted.

$\therefore x = 2$

At the turning point, $x = 2$, y = 3(2) + 6 = 12

Thus, the coordinates of point P are (2, 12).

(b) The straight line $y = 3x + 6$ intersects the y-axis at (0, 6).

The curve $y = 16 - x^2$ intersects the x-axis at (4, 0).

Area of the shaded region

= (Area under the straight line from x = 0 to x = 2) + (Area under the curve from x = 2 to x = 4)

$$= \frac{1}{2}(6+12)(2) + \int_2^4 \left(16 - x^2\right) dx$$

$$= 18 + \left[16x - \frac{x^3}{3}\right]_2^4$$

$$= 18 + \left[16(4) - \frac{4^3}{3} - \left((16)(2) - \frac{2^3}{3}\right)\right]$$

$$= 18 + \frac{128}{3} - \frac{88}{3}$$

$$= \frac{94}{3} \text{ units}^2$$

(c) The y-intercept of the curve $y = 16 - x^2$ is 16.

Generated volume

$$= \pi\int_{12}^{16} x^2\, dy$$

$$= \int_{12}^{16} \pi(16 - y)\, dy$$

$$= \pi\left[16y - \frac{y^2}{2}\right]_{12}^{16}$$

$$= \pi\left[16(16) - \frac{16^2}{2}\right] - \pi\left[16(12) - \frac{12^2}{2}\right]$$

$= 128\pi - 120\pi$
$= 8\pi$ units^3

UPSKILL 3.4

1 (a) $\frac{dT}{dt} = \frac{1}{2}(14 - t)$

At the turning point, T is a maximum,

$\frac{dT}{dt} = 0$

$\frac{1}{2}(14 - t) = 0$

$t = 14$

$\frac{d^2T}{dt^2} = -\frac{1}{2}$ (Negative)

Thus, T is a maximum.

(b) $\frac{dT}{dt} = \frac{1}{2}(14 - t)$

$T = \frac{1}{2}\int(14 - t)\, dt$

$T = \frac{1}{2}\left(14t - \frac{t^2}{2}\right) + c$

Given that $T = 25$ at the turning point where $t = 8$,

$25 = \frac{1}{2}\left(14(8) - \frac{8^2}{2}\right) + c$

$25 = 40 + c$

$c = -15$

Thus, $T = \frac{1}{2}\left(14t - \frac{t^2}{2}\right) - 15$

At the turning point where $t = 14$, the maximum value of T

$= \frac{1}{2}\left[14(14) - \frac{14^2}{2}\right] - 15$

$= 34$

2 (a) Sphere with a radius of 7 cm

(b) Volume of sphere

$$= \pi\int_{-7}^{7} x^2\, dy$$

$$= \pi\int_{-7}^{7} \left(49 - y^2\right) dy$$

$$= \pi\left[49y - \frac{y^3}{3}\right]_{-7}^{7}$$

$$= \pi\left[49(7) - \frac{7^3}{3} - \left((49)(-7) - \frac{(-7)^3}{3}\right)\right]$$

$$= \pi\left(\frac{686}{3} - \left(-\frac{686}{3}\right)\right)$$

$$= 457\frac{1}{3}\pi \text{ cm}^3$$

Summative Practice 3

1 $\int\left(t + \frac{1}{t}\right)\left(t - \frac{1}{t}\right) dt$

$= \int\left(t^2 - \frac{1}{t^2}\right) dt$

$= \int\left(t^2 - t^{-2}\right) dt$

$= \frac{t^3}{3} - \frac{t^{-1}}{-1} + c$

$= \frac{t^3}{3} + \frac{1}{t} + c$

2 $\int \frac{(1 + x)(1 - x)}{x^4}\, dx$

$= \int \frac{1 - x^2}{x^4}\, dx$

$= \int\left(x^{-4} - x^{-2}\right) dx$

$= \frac{x^{-3}}{-3} - \frac{x^{-1}}{-1} + c$

$= -\frac{1}{3x^3} + \frac{1}{x} + c$

3 $\int \frac{10}{(5 - 2y)^3}\, dy$

$= \int 10(5 - 2y)^{-3} dy$

$= 10\left[\frac{(5 - 2y)^{-2}}{-2(-2)}\right] + c$

$= \frac{5}{2(5 - 2y)^2} + c$

4 $\int \frac{3}{(1+2x)^4}\,dx$

$= \int 3(1+2x)^{-4}\,dx$

$= \left[\frac{3(1+2x)^{-3}}{-3(2)}\right] + c$

$= -\frac{1}{2}(1+2x)^{-3} + c$

But the given integral is $k(1+2x)^n + c$.

By comparison, $k = -\frac{1}{2}$ and $n = -3$.

5 $\frac{d}{dx}[f(x)] = 4g(x)$

$f(x) = 4\int g(x)\,dx$

$\int g(x)\,dx = \frac{1}{4}[f(x)] + c$

6 (a) $\frac{dy}{dx} = kx - 5$

$y = \int (kx-5)\,dx$

$y = \frac{kx^2}{2} - 5x + c$

The curve passes through the point (1, –1), thus $x = 1$ and $y = -1$.

$-1 = \frac{k}{2} - 5 + c$

$-2 = k - 10 + 2c$

$k + 2c = 8$ … (1)

The curve passes through the point (0, 3), thus $x = 0$ and $y = 3$.

$y = \frac{kx^2}{2} - 5x + c$

$3 = 0 - 0 + c$

$c = 3$

From (1) : $k + 2(3) = 8 \Rightarrow k = 2$

(b) $\frac{dy}{dx} = 2x - 5$

$y = \int (2x-5)\,dx$

$y = x^2 - 5x + h$

The curve passes through the point (0, 3), thus $x = 0$ and $y = 3$.

$3 = 0 - 0 + h$

$h = 3$

Thus, $y = x^2 - 5x + 3$

7 (a) $\frac{dy}{dx} = 2x + k$

$y = \int 2x + k$

$y = x^2 + kx + c$

The turning point is (1, 4).

$\frac{dy}{dx} = 2(1) + k = 0$

$k = -2$

(b) The curve passes through the point (1, 4), thus $x = 1$ and $y = 4$.

$y = x^2 - 2x + c$

$4 = 1^2 - 2(1) + c$

$c = 5$

Thus, the curve intersects the y-axis at the point (0, 5).

8 Gradient of normal = $\frac{1}{3-2x}$

Gradient of tangent = $-(3-2x)$.

$\frac{dy}{dx} = -3 + 2x$

$y = \int (-3+2x)\,dx$

$y = -3x + x^2 + c$

The curve passes through the point (3, 4), thus $x = 3$ and $y = 4$.

$4 = -3(3) + 3^2 + c$

$c = 4$

Hence , the equation of the curve is

$y = -3x + x^2 + 4$.

9 (a) Gradient of tangent

$= \frac{2-0}{4-2}$

$= 1$

(b) $\frac{dy}{dx} = 1$

$\frac{1}{2}x + p = 1$

$\frac{1}{2}(4) + p = 1$

$p = -1$

(c) $\frac{dy}{dx} = \frac{1}{2}x - 1$

$y = \int \left(\frac{1}{2}x - 1\right) dx$

$$y=\frac{1}{2}\left(\frac{x^2}{2}\right)-x+c$$

$$y=\frac{1}{4}x^2-x+c$$

The curve passes through the point (4, 2), thus $x = 4$ and $y = 2$.

$$2=\frac{1}{4}(4)^2-4+c$$

$$2=4-4+c$$

$$c = 2$$

Hence, the equation of the curve is

$$y=\frac{1}{4}x^2-x+2\,.$$

10 (a) The equation of the normal is $x+2y-5=0$.

$$x+2y-5=0$$

$$2y=-x+5$$

$$y=-\frac{1}{2}x+\frac{5}{2}$$

Since the gradient of the normal is $-\frac{1}{2}$, thus the gradient of the tangent is 2.

$$\frac{dy}{dx}=2$$

$$\frac{k}{x^2}+3=2$$

$$\frac{k}{1^2}+3=2$$

$$k=-1$$

(b) $\frac{dy}{dx}=-\frac{1}{x^2}+3$

$$y=\int\left(-\frac{1}{x^2}+3\right)dx$$

$$y=\int\left(-x^{-2}+3\right)dx$$

$$y=\frac{-x^{-1}}{-1}+3x+c$$

$$y=\frac{1}{x}+3x+c$$

The curve passes through the point (1, 2), thus $x = 1$ and $y = 2$.

$$y=\frac{1}{x}+3x+c$$

$$2=\frac{1}{1}+3(1)+c$$

$$c=-2$$

Hence, the equation of the curve is

$$y=\frac{1}{x}+3x-2\,.$$

11 (a) $\frac{dy}{dx}=2x+4$

$$y=\int\left(2x+4\right)dx$$

$$y=x^2+4x+c$$

Given $y=7$ at the turning point, $x = 1$,

$$7=1^2+4(1)+c$$

$$c=2$$

The equation of the curve is

$$y=x^2+4x+2\,.$$

(b) $y=x^2+4x+2$

$$\frac{dy}{dx}=2x+4$$

$$\frac{d^2y}{dx^2}=2$$

$$x^2\frac{d^2y}{dx^2}+(x-1)\frac{dy}{dx}+y+3=0$$

$$x^2(2)+(x-1)(2x+4)+x^2+4x+2+3=0$$

$$2x^2+2x^2+2x-4+x^2+4x+5=0$$

$$5x^2+6x+1=0$$

$$\left(5x+1\right)\left(x+1\right)=0$$

$$x=-\frac{1}{5}\text{ or }-1$$

12 (a) $\frac{dy}{dx}=3x^2-4x$

$$y=\int\left(3x^2-4x\right)dx$$

$$y=x^3-2x^2+c$$

The curve passes through the point (–1, –2), thus $x = -1$ and $y = -2$.

$$-2=(-1)^3-2(-1)^2+c$$

$$-2=-1-2+c$$

$$c=1$$

Hence, the equation of the curve is

$$y=x^3-2x^2+1\,.$$

(b) $y=x^3-2x^2+1$

$$\frac{dy}{dx}=3x^2-4x$$

$$\frac{d^2y}{dx^2}=6x-4$$

At the turning points,

$$\frac{dy}{dx}=0$$

$$3x^2-4x=0$$

$$x(3x-4)=0$$

$x=0$ or $x=\frac{4}{3}$

At the turning point, $x = 0$,

$y=0^3-2(0)^2+1=1$

Hence, (0, 1) is a turning point.

$\frac{d^2y}{dx^2}=6(0)-4=-2$ (Negative)

Hence, (0, 1) is a maximum point.

At the turning point, $x=\frac{4}{3}$.

$y=\left(\frac{4}{3}\right)^3-2\left(\frac{4}{3}\right)^2+1=-\frac{5}{27}$

Hence, $\left(\frac{4}{3},-\frac{5}{27}\right)$ is a turning point.

$\frac{d^2y}{dx^2}=6\left(\frac{4}{3}\right)-4=4$ (Positive)

Hence, $\left(\frac{4}{3},-\frac{5}{27}\right)$ is a minimum point.

13 (a) $\int_7^2 h(x)\,dx=-\int_2^7 h(x)\,dx$

$=-4$

(b) $\int_2^7 2h(x)\,dx-\int_2^7 p\,dx=6$

$2(4)-p[x]_2^7=6$

$8-p(7-2)=6$

$8-5p=6$

$-5p=-2$

$p=\frac{2}{5}$

14 (a) $y=6x-x^2$

At the turning point where $x = 2$,

$y=6(2)-2^2=8$

$\frac{dy}{dx}=6-2x=6-2(2)=2$

Hence, the equation of the tangent is

$y-8=2(x-2)$

$y-8=2x-4$

$y=2x+4$

(b) At the turning point, $x=0$, $y=4$

At the turning point, $x=6$,

$y=2(6)+4=16$

Area of the shaded region

= (Area of the trapezium from x = 0 to x = 6) – (Area under the curve from x = 0 to x = 6)

$=\frac{1}{2}(4+16)(6)-\int_0^6 (6x-x^2)\,dx$

$=60-\left[3x^2-\frac{x^3}{3}\right]_0^6$

$=60-\left[3(6)^2-\frac{6^3}{3}\right]$

$=60-(108-72)$

$=60-36$

$=24$ units^2

15 (a) At the x-axis, $y = 0$

$y=27-(x-2)^3$

$0=27-(x-2)^3$

$(x-2)^3=27$

$x-2=3$

$x=5$

Thus, P is point (5, 0).

(b) Area of the shaded region

$=\int_0^5 \left[27-(x-2)^3\right]dx$

$=\left[27x-\frac{(x-2)^4}{4(1)}\right]_0^5$

$=\left[27(5)-\frac{3^4}{4}-\left(27(0)-\frac{(-2)^4}{4}\right)\right]$

$=\frac{459}{4}-\left(-\frac{16}{4}\right)$

$=\frac{475}{4}$ units^2

16 $y=2x$ … (1)

$y=6x-x^2$ … (2)

Substitute (1) into (2) :

$2x=6x-x^2$

$x^2-6x+2x=0$

$x^2-4x=0$

$x(x-4)=0$

$x=0$ or 4

At the turning point, x = 0, y = 0

At the turning point, x = 4, y = 2(4) = 8

Thus, the points of intersection between the straight line $y = 2x$ and the curve $y = 6x - x^2$ are (0, 0) and (4, 8).

Area of the shaded region B
= (Area of triangle from x = 0 to x = 4) + (Area under of curve from x = 4 to x = 6)

$= \left(\frac{1}{2} \times 4 \times 8\right) + \int_4^6 \left(6x - x^2\right) dx$

$= 16 + \left[3x^2 - \frac{x^3}{3}\right]_4^6$

$= 16 + \left[3(6)^2 - \frac{6^3}{3} - \left(3(4)^2 - \frac{4^3}{3}\right)\right]$

$= 16 + \left[36 - \frac{80}{3}\right]$

$= 16 + \frac{28}{3}$

$= \frac{76}{3}$ units^2

Area under of curve from x = 0 to x = 6

$= \int_0^6 \left(6x - x^2\right) dx$

$= \left[3x^2 - \frac{x^3}{3}\right]_0^6$

$= 3(6)^2 - \frac{6^3}{3} - 0$

$= 36$ units^2

Area of the shaded region A

$= 36 - \frac{76}{3}$

$= \frac{32}{3}$ units^2

Area of shaded region A : Area of shaded region B

$= \frac{\text{Area of region } A}{\text{Area of region } B}$

$= \frac{\frac{32}{3}}{\frac{76}{3}}$

$= \frac{8}{19}$

= 8 : 19 [Shown]

17 (a) Area of region A

$= \int_1^2 y\,dx$

$= \int_1^2 \frac{8}{x^2}\,dx$

$= \int_1^2 8x^{-2}\,dx$

$= \left[\frac{8x^{-1}}{-1}\right]_1^2$

$= \left[-\frac{8}{x}\right]_1^2$

$= -\frac{8}{2} - \left(\frac{-8}{1}\right)$

$= -4 + 8$

= 4 units^2

(b) Area of region B = Area of region C

$\int_2^p 8x^{-2}\,dx = \int_p^5 8x^{-2}\,dx$

$\left[-\frac{8}{x}\right]_2^p = \left[-\frac{8}{x}\right]_p^5$

$-\frac{8}{p} - \left(-\frac{8}{2}\right) = -\frac{8}{5} - \left(-\frac{8}{p}\right)$

$-\frac{8}{p} + 4 = -\frac{8}{5} + \frac{8}{p}$

$\frac{16}{p} = \frac{28}{5}$

$p = \frac{5}{28} \times 16$

$p = 2\frac{6}{7}$

18 (a) $y = (x-1)^2 = x^2 - 2x + 1$

$\frac{dy}{dx} = 2x - 2$

At the point $P(3, 4)$,

$m = \frac{dy}{dx} = 2(3) - 2 = 4$

Equation of the tangent is

$y - 4 = 4(x - 3)$

$y - 4 = 4x - 12$

$y = 4x - 8$

At point Q (x-axis),

$y = 4x - 8$

$0 = 4x - 8$

$4x = 8$

$x = 2$

Thus, Q is point (2, 0).

(b) Area of the shaded region
= (Area under the curve from x = 1 to x = 3) – (Area of triangle from x = 2 to x = 3)

$= \int_1^3 (x-1)^2\,dx - \left(\frac{1}{2}\times 1\times 4\right)$

$= \left[\frac{(x-1)^3}{3(1)}\right]_1^3 - 2$

$= \frac{2^3}{3} - 0 - 2$

$= \frac{2}{3}$ unit^2

19 (a) $y = x^3 - 3x + c$

$\frac{dy}{dx} = 3x^2 - 3$

At the minimum point P,

$\frac{dy}{dx} = 0$

$3x^2 - 3 = 0$

$3x^2 = 3$

$x^2 = 1$

$x = 1$

Thus, P is point (1, 0).

Substitute x = 1 and y =0 into $y = x^3 - 3x + c$.

$0 = 1^3 - 3(1) + c$

$c = 2$

(b) Area of the shaded region

$= \int_0^1 \left(x^3 - 3x + 2\right) dx$

$= \left[\frac{x^4}{4} - \frac{3x^2}{2} + 2x\right]_0^1$

$= \frac{1}{4} - \frac{3}{2} + 2 - 0$

$= \frac{3}{4}$ unit^2

20 $y = -x + 5$ … (1)

$y = x^2 - 2x + 5$ … (2)

Substitute (2) into (1) :

$x^2 - 2x + 5 = -x + 5$

$x^2 - x = 0$

$x(x-1) = 0$

$x = 0$ or $x = 1$

The straight line $y = -x + 5$ intersects the x-axis at (5, 0).

At the turning point, x = 0, $y = -0 + 5 = 5$

At the turning point, $x = 1$, $y = -1 + 5 = 4$

Thus, the points of intersection between the curve and the straight line are (0, 5) and (1, 4).

Area of the shaded region
= (Area under the curve from x = 0 to x = 1) + (Area of triangle from x = 1 to x = 5)

$= \int_0^1 \left(x^2 - 2x + 5\right) dx + \frac{1}{2}\times(5-1)\times 4$

$= \left[\frac{x^3}{3} - x^2 + 5x\right]_0^1 + 8$

$= \left(\frac{1}{3} - 1 + 5\right) + 8$

$= \frac{37}{3}$ units^2

21 $x = 3$ … (1)

$y^2 = 4(x+1)$ … (2)

Substitute (1) into (2) :

$y^2 = 4(3+1) = 16$

$y = 4$

Thus, B is point (3, 4) and A is point (0, 4).

$y^2 = 4(x+1)$

At the y-axis, x = 0.

$y^2 = 4(0+1)$

$y = 2$

Thus, the curve intersects the y-axis at (0, 2).

Area of the shaded region

$= \int_2^4 x\,dy$

$y^2 = 4(x+1)$

$\frac{y^2}{4} = x + 1$

$x = \frac{y^2}{4} - 1$

$= \int_2^4 \left(\frac{y^2}{4} - 1\right) dy$

$= \left[\frac{y^3}{12} - y\right]_2^4$

$= \frac{4^3}{12} - 4 - \left(\frac{2^3}{12} - 2\right)$

$= \frac{4}{3} - \left(-\frac{4}{3}\right)$

$= \frac{8}{3}$ units^2

22 (a) Point (k, 3) lies on the straight line $y = 5 - x$.
Thus, $3 = 5 - k \Rightarrow k = 2$.

(b) The straight line $y = 5 - x$ intersects the y-axis at (0, 5).

The curve $y = (1 - x)(x - 5)$ intersects the x-axis (1, 0) and (5, 0).

Area of the shaded region
= (Area of trapezium from x = 0 to x = 2) – Area under the curve from x = 1 to x = 2)

$= \frac{1}{2}(5+3)(2) - \int_1^2 y\, dx$

$= 8 - \int_1^2 (1-x)(x-5)\, dx$

$= 8 - \int_1^2 \left(-x^2 + 6x - 5\right) dx$

$= 8 - \left[-\frac{x^3}{3} + 3x^2 - 5x\right]_1^2$

$= 8 - \left[-\frac{2^3}{3} + 3(2)^2 - 5(2) - \left(-\frac{1}{3} + 3 - 5\right)\right]$

$= 8 - \left[-\frac{2}{3} - \left(-\frac{7}{3}\right)\right]$

$= \frac{19}{3}$ units^2

23 Area of the shaded region

$= \int_{-2}^{2} \left[\left(-x^2 + 2x + 7\right) - \left(x^2 + 2x - 1\right)\right] dx$

$= \int_{-2}^{2} \left(-2x^2 + 8\right) dx$

$= \left[-\frac{2x^3}{3} + 8x\right]_{-2}^{2}$

$= \left[-\frac{2(2)^3}{3} + 8(2) - \left(-\frac{2}{3}(-2)^3 + 8(-2)\right)\right]$

$= \frac{32}{3} - \left(-\frac{32}{3}\right)$

$= 21\frac{1}{3}$ units^2

24 $y = x(x-2) = x^2 - 2x$

$y^2 = \left(x^2 - 2x\right)^2 = \left(x^4 - 4x^3 + 4x^2\right)$

Generated volume

$= \pi \int_0^3 y^2\, dx$

$= \pi \int_0^3 \left(x^4 - 4x^3 + 4x^2\right) dx$

$= \pi \left[\frac{x^5}{5} - \frac{4x^4}{4} + \frac{4x^3}{3}\right]_0^3$

$= \pi \left[\frac{x^5}{5} - x^4 + \frac{4x^3}{3}\right]_0^3$

$= \pi \left(\frac{3^5}{5} - 3^4 + 36 - 0\right)$

$= \pi \left(\frac{243}{5} - 81 + 36\right)$

$= \frac{18}{5}\pi$ units^3

25 Generated volume

$= \pi \int_{-2}^{1} x^2\, dy$

$= \pi \int_{-2}^{1} \left(y^2 + 4\right)^2 dy$

$= \pi \int_{-2}^{1} \left(y^4 + 8y^2 + 16\right) dy$

$= \pi \left[\frac{y^5}{5} + \frac{8y^3}{3} + 16y\right]_{-2}^{1}$

$= \pi \left[\frac{1}{5} + \frac{8}{3} + 16 - \left(-\frac{32}{5} + \frac{8}{3}(-8) - 32\right)\right]$

$= \pi \left[\frac{283}{15} - \left(-\frac{896}{15}\right)\right]$

$= \frac{393}{5}\pi$ units^3

26 Generated volume

$= \pi \int_2^5 x^2\, dy$

$= \pi \int_2^5 \left(25 - y^2\right) dy$

$= \pi \left[25y - \frac{y^3}{3}\right]_2^5$

$$= \pi\left[25(5) - \frac{5^3}{3} - \left(25(2) - \frac{2^3}{3}\right)\right]$$
$$= \pi\left[125 - \frac{125}{3} - \left(50 - \frac{8}{3}\right)\right]$$
$$= \pi\left(\frac{250}{3} - \frac{142}{3}\right)$$
$$= 36\pi \text{ units}^3$$

27 (a) Generated volume

$$= \pi\int_0^4 x^2\, dy$$
$$= \pi\int_0^4 (4 - y)\, dy$$
$$= \pi\left[4y - \frac{y^2}{2}\right]_0^4$$
$$= \pi(16 - 8 - 0)$$
$$= 8\pi \text{ units}^3$$

(b) The curve intersects the x-axis at the points (–2, 0) and (2, 0).

Generated volume

$$= 2\pi\int_0^2 y^2\, dx$$
$$= 2\pi\int_0^2 \left(4 - x^2\right)^2\, dx$$
$$= 2\pi\int_0^2 \left(16 - 8x^2 + x^4\right)\, dx$$
$$= 2\pi\left[16x - \frac{8x^3}{3} + \frac{x^5}{5}\right]_0^2$$
$$= 2\pi\left(16(2) - \frac{8(2)^3}{3} + \frac{2^5}{5} - 0\right)$$
$$= 2\pi\left(32 - \frac{64}{3} + \frac{32}{5} - 0\right)$$
$$= \frac{512}{15}\pi \text{ units}^3$$

28 (a) The curve intersects the x-axis at (2, 0) and the y-axis at the point (0, 5).

Generated volume

$$= \pi\int_0^4 y^2\, dx$$
$$= \pi\int_0^4 \frac{16 - x^2}{4}\, dx$$
$$= \frac{1}{4}\pi\int_0^4 \left(16 - x^2\right) dx$$
$$= \frac{1}{4}\pi\left[16x - \frac{x^3}{3}\right]_0^4$$
$$= \frac{1}{4}\pi\left[16(4) - \frac{4^3}{3} - 0\right]$$
$$= \frac{1}{4}\pi\left(\frac{128}{3}\right)$$
$$= \frac{32}{3}\pi \text{ units}^2$$

(b) Generated volume

$$= \pi\int_0^4 x^2\, dy$$
$$= \pi\int_0^2 \left(16 - 4y^2\right) dy$$
$$= \pi\left[16y - \frac{4y^3}{3}\right]_0^2$$
$$= \pi\left(32 - \frac{32}{3}\right)$$
$$= \frac{64}{3}\pi \text{ units}^3$$

29 $y = 3\sqrt{x - 1}$

$y^2 = 9(x - 1)$

At the x-axis, $y = 0$

$0^2 = 9(x - 1)$

$x = 1$

Generated volume = $\frac{9}{2}\pi$ units^3

$$\pi\int_1^k y^2\, dx = \frac{9}{2}\pi$$
$$\int_1^k 9(x - 1)\, dx = \frac{9}{2}$$
$$\int_1^k (x - 1)\, dx = \frac{1}{2}$$
$$\left[\frac{x^2}{2} - x\right]_1^k = \frac{1}{2}$$
$$\frac{k^2}{2} - k - \left(\frac{1}{2} - 1\right) = \frac{1}{2}$$
$$\frac{k^2}{2} - k + \frac{1}{2} = \frac{1}{2}$$
$$\frac{k^2}{2} - k = 0$$
$$k^2 - 2k = 0$$
$$k(k - 2) = 0$$
$$k = 2$$

30 Generated volume $= 2\pi$

$$\pi\int_{-3}^{k} x^2\, dy = 2\pi$$

$$\int_{-3}^{k} (y+3)\, dy = 2$$

$$\left[\frac{y^2}{2} + 3y\right]_{-3}^{k} = 2$$

$$\frac{k^2}{2} + 3k - \left(\frac{9}{2} - 9\right) = 2$$

$$k^2 + 6k - (9 - 18) = 4$$

$$k^2 + 6k + 5 = 0$$

$$(k+1)(k+5) = 0$$

$$k = -5 \text{ or } k = -1$$

$k = -5$ is not accepted.

$\therefore k = -1$

31 (a) $3y = x$ … (1)

$x^2 + 7y^2 = 16$ … (2)

Substitute (1) into (2) :

$$(3y)^2 + 7y^2 = 16$$

$$16y^2 = 16$$

$$y^2 = 1$$

$$y = \pm 1$$

$y = -1$ is not accepted.

$\therefore y = 1$

From (1) :

$3(1) = x \Rightarrow x = 3$

Thus, A is point (3, 1).

$x^2 + 7y^2 = 16$

At the x-axis, $y = 0$.

$x^2 + 7(0)^2 = 16$

$x = 4$

Thus, B is point (4, 0).

(b) $x^2 + 7y^2 = 16$

$$y^2 = \frac{16 - x^2}{7}$$

Generated volume

= (Volume of cone generated by the straight line from $x = 0$ to $x = 3$) + Volume generated by the curve from $x = 3$ to $x = 4$)

$$= \frac{1}{3}\pi r^2 h + \pi\int_{3}^{4} y^2\, dx$$

$$= \frac{1}{3}\pi(1)(3) + \pi\int_{3}^{4}\left(\frac{16 - x^2}{7}\right) dx$$

$$= \pi + \frac{1}{7}\pi\left[16x - \frac{x^3}{3}\right]_{3}^{4}$$

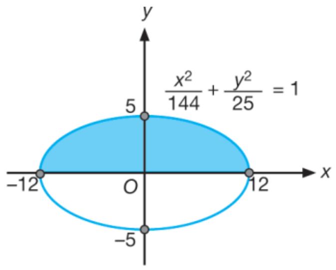
$$= \pi + \frac{1}{7}\pi\left[16(4) - \frac{64}{3} - \left(48 - \frac{27}{3}\right)\right]$$

$$= \pi + \frac{1}{7}\pi\left[16(4) - \frac{64}{3} - \left(48 - \frac{27}{3}\right)\right]$$

$$= \pi + \frac{1}{7}\left(\frac{11}{3}\right)\pi$$

$$= \frac{32}{21}\pi \text{ units}^3$$

32



Smart Strategy

The volume of the rugby ball is given by the generated volume when the shaded region is rotated through 360° about the x-axis.

HOT TIPS

$\frac{x^2}{144} + \frac{y^2}{25} = 1$ is the equation of an ellipse.

$$\frac{x^2}{144} + \frac{y^2}{25} = 1$$

$$\frac{y^2}{25} = 1 - \frac{x^2}{144}$$

$$y^2 = 25\left(1 - \frac{x^2}{144}\right)$$

Volume of the rugby ball

= Generated volume

$$= \pi\int_{-12}^{12} y^2\, dx$$

$$= \pi\int_{-12}^{12} 25\left(1 - \frac{x^2}{144}\right) dx$$

$$= 25\pi\left[x - \frac{x^3}{432}\right]_{-12}^{12}$$

$$= 25\pi\left[12-\frac{12^3}{432}-\left[-12-\frac{(-12)^3}{432}\right]\right]$$
$$= 25\pi[8-(-8)]$$
$$= 400\pi \text{ units}^3$$
$$= 400\times 3.142$$
$$= 1\ 256.8 \text{ units}^3$$

Hence, the volume of the rugby ball is 1 256.8 cm^3.

SPM Spot

1 $\frac{dy}{dx}=(2x+1)^3$

$y=\int (2x+1)^3\, dx$

$y=\frac{(2x+1)^4}{4(2)}+c$

$y=\frac{(2x+1)^4}{8}+c$

The curve passes through the point $\left(\frac{1}{2},-3\right)$.

$-3=\frac{\left[2\left(\frac{1}{2}\right)+1\right]^4}{8}+c$

$-3=2+c$

$c=-5$

Hence, the equation of the curve is

$y=\frac{(2x+1)^4}{8}-5$.

2 (a) (i) $y=x^2-6x+16\ \ldots (1)$

$y=6x-x^2\ \ldots (2)$

Substitute (1) into (2) :

$x^2-6x+16=6x-x^2$

$2x^2-12x+16=0$

$x^2-6x+8=0$

$(x-2)(x-4)=0$

$x=2$ or $x=4$

From (2):

When $x = 2$, $y=6(2)-2^2=8$

When $x = 4$, $y=6(4)-4^2=8$

Hence, A is point (2, 8) and B is point (4, 8).

(ii) Area bounded by the two curves

$$=\int_2^4 (6x-x^2)\,dx-\int_2^4 (x^2-6x+16)\,dx$$
$$=\int_2^4 (6x-x^2-x^2+6x-16)\,dx$$
$$=\int_2^4 (12x-2x^2-16)\,dx$$
$$=\left[6x^2-\frac{2x^3}{3}-16x\right]_2^4$$
$$=6(4)^2-\frac{2}{3}(4)^3-16(4)-\left[6(2)^2-\frac{2}{3}(2)^3-16(2)\right]$$
$$=-10\frac{2}{3}-\left(-13\frac{1}{3}\right)$$
$$=2\frac{2}{3}\text{ units}^2$$

(b) Generated volume $=42\frac{1}{2}\pi$

$\pi\int_3^k y^2\,dx$ – Volume of cone $=42\frac{1}{2}\pi$

$\pi\int_3^k (x+6)\,dx-\frac{1}{3}\pi r^2 h=\frac{85}{2}\pi$

$\left[\frac{x^2}{2}+6x\right]_3^k-\frac{1}{3}(3)^2(k-3)=\frac{85}{2}$

$\frac{k^2}{2}+6k-\left(\frac{9}{2}+18\right)-3k+9=\frac{85}{2}$

$\frac{k^2}{2}+3k-56=0$

$k^2+6k-112=0$

$(k-8)(k+14)=0$

$k=8$ or $k=-14$

$k=-14$ is not accepted.

$\therefore k=8$

3 (a) $y^2=-x+9$

$y=\sqrt{-x+9}$

$y=(-x+9)^{\frac{1}{2}}$

$\frac{dy}{dx}=\frac{1}{2}(-x+9)^{-\frac{1}{2}}(-1)$

$=-\frac{1}{2\sqrt{-x+9}}$

When $x = 5$, $\frac{dy}{dx}=-\frac{1}{2\sqrt{-5+9}}=-\frac{1}{4}$

Thus, the gradient of the tangent = $-\frac{1}{4}$

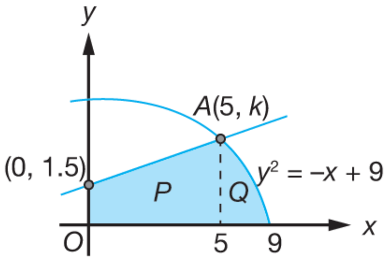
Equation of the tangent is

$$y-2=-\frac{1}{4}(x-5)$$

$y=\sqrt{-5+9}=2$
Thus, $k = 2$

$$4y-8=-x+5$$
$$4y=-x+13$$

(b)



Area of the shaded region
= Area P + Area Q

$$= \frac{1}{2}(2+1.5)(5) + \int_5^9 (-x+9)^{\frac{1}{2}}\, dx +$$

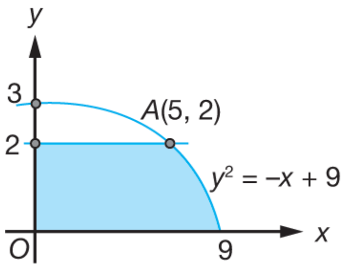
$$= 8\frac{3}{4} + \left[\frac{(-x+9)^{\frac{3}{2}}}{-\frac{3}{2}}\right]_5^9$$

$$= 8\frac{3}{4} - \frac{2}{3}\left[(-9+9)^{\frac{3}{2}} - (-5+9)^{\frac{3}{2}}\right]$$

$$= 8\frac{3}{4} - \frac{2}{3}[0-8]$$

$$= 14\frac{1}{12} \text{ units}^2$$

(c)



$$V_y = \pi\int_0^2 x^2\, dy$$

$$= \pi\int_0^2 \left(-y^2+9\right)^2\, dy$$

$$= \pi\int_0^2 \left(y^4-18y^2+81\right) dy$$

$$= \pi\left[\frac{y^5}{5} - 18\times\frac{y^3}{3} + 81y\right]_0^2$$

$$= \pi\left[\frac{2^5}{5} - 6(2)^3 + 81(2) - 0\right]$$

$$= 120\frac{2}{5}\pi \text{ units}^3$$