Form 5 Chapter 2 Differentiation Fully-Worked Solutions

UPSKILL 2.1a

1 (a) $\lim_{n\to 0}\left(\frac{n^2+n}{n}\right)$

$= \lim_{n\to 0}\left(\frac{n(n+1)}{n}\right)$

$= \lim_{n\to 0}(n+1)$

$= 0+1$

$= 1$

(b) $\lim_{n\to 7}\left(\frac{n^2-49}{n-7}\right)$

$= \lim_{n\to 7}\left(\frac{(n+7)(n-7)}{n-7}\right)$

$= \lim_{n\to 7}(n+7)$

$= 7+7$

$= 14$

(c) $\lim_{n\to 1}\left(\frac{n^2+n-2}{n-1}\right)$

$= \lim_{n\to 1}\left(\frac{(n+2)(n-1)}{n-1}\right)$

$= \lim_{n\to 1}(n+2)$

$= 1+2$

$= 3$

(d) $\lim_{n\to 2}\left(\frac{n^2-n-2}{n^2-5n+6}\right)$

$= \lim_{n\to 2}\left(\frac{(n-2)(n+1)}{(n-3)(n-2)}\right)$

$= \lim_{n\to 2}\left(\frac{n+1}{n-3}\right)$

$= \frac{2+1}{2-3}$

$= -3$

UPSKILL 2.1b

1 (a) $y = x^2 - 2 \ \ldots (1)$

$y+\delta y = (x+\delta x)^2 - 2$

$y+\delta y = x^2 + 2x\delta x + (\delta x)^2 - 2 \ \ldots (2)$

$(2)-(1):$

$\delta y = 2x\delta x + (\delta x)^2$

$\frac{\delta y}{\delta x} = 2x+\delta x$

$\frac{dy}{dx} = \lim_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\right)$

$\frac{dy}{dx} = \lim_{\delta x\to 0}(2x+\delta x)$

$\frac{dy}{dx} = 2x+0$

$\frac{dy}{dx} = 2x$

(b) $y = x^2 + x - 6 \ \ldots (1)$

$y+\delta y = (x+\delta x)^2 + x + \delta x - 6$

$y+\delta y = x^2 + 2x\delta x + (\delta x)^2 + x + \delta x - 6 \ \ldots (2)$

$(2)-(1):\ \delta y = 2x\delta x + (\delta x)^2 + \delta x$

$\frac{\delta y}{\delta x} = 2x+\delta x+1$

$\frac{dy}{dx} = \lim_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\right)$

$\frac{dy}{dx} = \lim_{\delta x\to 0}(2x+\delta x+1)$

$\frac{dy}{dx} = 2x+0+1$

$\frac{dy}{dx} = 2x+1$

2 (a) $y = \frac{3}{x} \ \ldots (1)$

$y+\delta y = \frac{3}{x+\delta x} \ \ldots (2)$

$(2)-(1):$

$\delta y = \frac{3}{x+\delta x} - \frac{3}{x}$

$\delta y = \frac{3x-3(x+\delta x)}{(x+\delta x)^2}$

$\delta y = \frac{-3\delta x}{(x+\delta x)^2}$

$\frac{\delta y}{\delta x} = \frac{-3}{(x+\delta x)^2}$

$\frac{dy}{dx} = \lim_{\delta x\to 0}\left(\frac{\delta y}{\delta x}\right)$

$\frac{dy}{dx} = \lim_{\delta x\to 0}\frac{-3}{(x+\delta x)^2}$

$$\frac{dy}{dx} = \frac{-3}{(x+0)^2}$$
$$\frac{dy}{dx} = \frac{-3}{x^2}$$

(b) $$y = \frac{5}{x^2} \quad \ldots (1)$$
$$y + \delta y = \frac{5}{(x+\delta x)^2} \quad \ldots (2)$$

(2) – (1) :
$$\delta y = \frac{5}{(x+\delta x)^2} - \frac{5}{x^2}$$
$$\delta y = \frac{5x^2 - 5\left[(x+\delta x)^2\right]}{x^2(x+\delta x)^2}$$
$$\delta y = \frac{5x^2 - 5\left[x^2 + 2x\,\delta x + (\delta x)^2\right]}{x^2(x+\delta x)^2}$$
$$\delta y = \frac{5x^2 - 5x^2 - 10x\delta x - 5(\delta x)^2}{x^2(x+\delta x)^2}$$
$$\delta y = \frac{-10x\delta x - 5(\delta x)^2}{x^2(x+\delta x)^2}$$
$$\frac{\delta y}{\delta x} = \frac{-10x - 5\delta x}{x^2(x+\delta x)^2}$$
$$\frac{dy}{dx} = \lim_{\delta x \to 0}\left(\frac{\delta y}{\delta x}\right)$$
$$\frac{dy}{dx} = \lim_{\delta x \to 0}\left(\frac{-10x - 5\delta x}{x^2(x+\delta x)^2}\right)$$
$$\frac{dy}{dx} = \frac{-10x - 5(0)}{x^2(x+0)^2}$$
$$\frac{dy}{dx} = \frac{-10x}{x^2(x)^2}$$
$$\frac{dy}{dx} = \frac{-10}{x^3}$$

UPSKILL 2.2a

1 (a) $y = 24$
$$\frac{dy}{dx} = 0$$

(b) $y = 6x$
$$\frac{dy}{dx} = 6$$

(c) $y = x^8$
$$\frac{dy}{dx} = 8x^7$$

(d) $y = 5x^4$
$$\frac{dy}{dx} = 20x^3$$

(e) $y = \frac{4}{x}$
$$\frac{dy}{dx} = -\frac{4}{x^2}$$

(f) $y = -\frac{4}{x^3}$
$$\frac{dy}{dx} = \frac{12}{x^4}$$

(g) $y = \frac{x^5}{15}$
$$\frac{dy}{dx} = \frac{1}{15}\left(5x^4\right) = \frac{x^4}{3}$$

(h) $y = -\frac{3}{4x^8} = -\frac{3}{4}\left(x^{-8}\right)$
$$\frac{dy}{dx} = -\frac{3}{4}\left(-8x^{-9}\right) = 6x^{-9} = \frac{6}{x^9}$$

2 (a) $\frac{d}{dx}\left(x^6 - 3x^3 + 5\right) = 6x^5 - 9x^2$

(b) $\frac{d}{dx}\left(\frac{1}{2}t^4 + 3t^2 + 6\right) = 2t^3 + 6t$

(c) $\frac{d}{dj}\left(5j^3 - 4j^2 - 3j\right) = 15j^2 - 8j - 3$

3 (a) $f(x) = 3x^2 + \frac{3}{x} + \frac{3}{x^2}$
$$f'(x) = 6x - \frac{3}{x^2} - \frac{6}{x^3}$$

(b) $f(x) = 2x + \frac{2}{x} - \frac{2}{x^3}$
$$f'(x) = 2 - \frac{2}{x^2} + \frac{6}{x^4}$$

(c) $f(x) = \frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3}$
$$f'(x) = -\frac{1}{x^2} - \frac{2}{x^3} - \frac{3}{x^4}$$

4 (a) $s=(t-1)(2t+5)$

$s=2t^2+3t-5$

$\frac{ds}{dt}=4t+3$

(b) $s=(2t-5)^2=4t^2-20t+25$

$\frac{ds}{dt}=8t-20$

(c) $s=3t^2(t-2)^2$

$s=3t^2\left(t^2-4t+4\right)$

$s=3t^4-12t^3+12t^2$

$\frac{ds}{dt}=12t^3-36t^2+24t$

(d) $s=\left(t-\frac{2}{t}\right)^2=t^2-4+\frac{4}{t^2}$

$\frac{ds}{dt}=2t-\frac{8}{t^3}$

5 (a) $v=\frac{2t^2-9}{t}=2t-\frac{9}{t}$

$\frac{dv}{dt}=2+\frac{9}{t^2}$

(b) $v=\frac{t^2+6t-12}{t^2}=1+\frac{6}{t}-\frac{12}{t^2}$

$\frac{dv}{dt}=-\frac{6}{t^2}+\frac{24}{t^3}$

(c) $v=\frac{(t+1)(2-3t)}{t^2}=\frac{2-t-3t^2}{t^2}$

$v=\frac{2}{t^2}-\frac{1}{t}-3$

$\frac{dv}{dt}=-\frac{4}{t^3}+\frac{1}{t^2}$

UPSKILL 2.2b

1 (a) $y=2x^3\left(2-x^5\right)=4x^3-2x^8$

$\frac{dy}{dx}=12x^2-16x^7$

(b) $y=\left(6+x^2\right)(5-3x)$

$=30-18x+5x^2-3x^3$

$\frac{dy}{dx}=-9x^2+10x-18$

(c) $y=\left(5x^2-4\right)(3-x)$

$=15x^2-5x^3-12+4x$

$\frac{dy}{dx}=-15x^2+30x+4$

(d) $y=\left(x^4+1\right)\left(x^3-2x\right)$

$=x^7-2x^5+x^3-2x$

$\frac{dy}{dx}=7x^6-10x^4+3x^2-2$

(e) $y=(2x-1)\left(x^2+3x-2\right)$

$\frac{dy}{dx}=(2x-1)(2x+3)+\left(x^2+3x-2\right)(2)$

$=4x^2+4x-3+2x^2+6x-4$

$=6x^2+10x-7$

(f) $y=\left(2x^2+1\right)\left(3x^2+x-4\right)$

$\frac{dy}{dx}=(2x^2+1)(6x+1)+(3x^2+x-4)(4x)$

$=12x^3+2x^2+6x+1+$

$12x^3+4x^2-16x$

$=24x^3+6x^2-10x+1$

(g) $y=\left(\frac{1}{x}+1\right)\left(\frac{2}{x^2}-3\right)$

$y=\frac{2}{x^3}-\frac{3}{x}+\frac{2}{x^2}-3$

$\frac{dy}{dx}=-\frac{6}{x^4}+\frac{3}{x^2}-\frac{4}{x^3}$

(h) $y=\left(2x^2+x^3\right)\left(x-\frac{2}{x}\right)$

$=2x^3-4x+x^4-2x^2$

$\frac{dy}{dx}=6x^2-4+4x^3-4x$

2 (a) $f(x)=x\left(x^2+4x-3\right)$

$=x^3+4x^2-3x$

$f'(x)=3x^2+8x-3$

(b) $f'(x)=0$

$3x^2+8x-3=0$

$x=\frac{1}{3}$ or -3

3 (a) $y = \frac{5x}{x+3}$

$\frac{dy}{dx} = \frac{5(x+3)-5x}{(x+3)^2}$

$= \frac{5x+15-5x}{(x+3)^2}$

$= \frac{15}{(x+3)^2}$

(b) $y = \frac{6x^2}{x-5}$

$\frac{dy}{dx} = \frac{(x-5)(12x)-6x^2}{(x-5)^2}$

$= \frac{12x^2-60x-6x^2}{(x-5)^2}$

$= \frac{6x^2-60x}{(x-5)^2}$

(c) $y = \frac{2x-5}{2x+3}$

$\frac{dy}{dx} = \frac{(2x+3)(2)-(2x-5)(2)}{(2x+3)^2}$

$= \frac{4x+6-4x+10}{(2x+3)^2}$

$= \frac{16}{(2x+3)^2}$

(d) $y = \frac{x^2-5}{x+3}$

$\frac{dy}{dx} = \frac{(x+3)(2x)-(x^2-5)}{(x+3)^2}$

$= \frac{2x^2+6x-x^2+5}{(x+3)^2}$

$= \frac{x^2+6x+5}{(x+3)^2}$

(e) $y = \frac{4x^2}{7-x^2}$

$\frac{dy}{dx} = \frac{(7-x^2)(8x)-4x^2(-2x)}{(7-x^2)^2}$

$= \frac{56x-8x^3+8x^3}{(7-x^2)^2}$

$= \frac{56x}{(7-x^2)^2}$

(f) $y = \frac{x^2}{x^2-5}$

$\frac{dy}{dx} = \frac{(x^2-5)(2x)-(x^2)(2x)}{(x^2-5)^2}$

$= \frac{2x^3-10x-2x^3}{(x^2-5)^2}$

$= \frac{-10x}{(x^2-5)^2}$

(g) $y = \frac{3x^4}{9-x^2}$

$\frac{dy}{dx} = \frac{(9-x^2)(12x^3)-3x^4(-2x)}{(9-x^2)^2}$

$= \frac{108x^3-12x^5+6x^5}{(9-x^2)^2}$

$= \frac{108x^3-6x^5}{(9-x^2)^2}$

(h) $y = \frac{x^2-2x}{x^2+3x}$

$\frac{dy}{dx} = \frac{(x^2+3x)(2x-2)-(x^2-2x)(2x+3)}{(x^2+3x)^2}$

$= \frac{2x^3-2x^2+6x^2-6x-\left[2x^3+3x^2-4x^2-6x\right]}{(x^2+3x)^2}$

$= \frac{2x^3+4x^2-6x-\left[2x^3-x^2-6x\right]}{(x^2+3x)^2}$

$= \frac{5x^2}{(x^2+3x)^2}$

4 (a) $f(x) = \frac{2-x}{x^2+5}$

$f'(x) = \frac{(x^2+5)(-1)-(2-x)(2x)}{(x^2+5)^2}$

$= \frac{-x^2-5-4x+2x^2}{(x^2+5)^2}$

$= \frac{x^2-4x-5}{(x^2+5)^2}$

(b) $f'(x) = 0$

$x^2-4x-5=0$

$x = 5$ or -1

5 (a) $y=(4x+7)^5$

$\frac{dy}{dx}=5(4x+7)^4(4)=20(4x+7)^4$

(b) $y=\left(6-x^2\right)^4$

$\frac{dy}{dx}=4(6-x^2)^3(-2x)=-8x(6-x^2)^3$

(c) $y=\left(\frac{1}{3}x^3-9\right)^6$

$\frac{dy}{dx}=6\left(\frac{x^3}{3}-9\right)^5\left(x^2\right)=6x^2\left(\frac{x^3}{3}-9\right)^5$

(d) $y=\left(x^2+5x-3\right)^3$

$\frac{dy}{dx}=3\left(x^2+5x-3\right)^2(2x+5)$

6 (a) $s=\frac{2}{5t+1}=2(5t+1)^{-1}$

$\frac{ds}{dt}=-2(5t+1)^{-2}(5)$

$=-\frac{10}{(5t+1)^2}$

(b) $s=\frac{3}{t^2+4}=3\left(t^2+4\right)^{-1}$

$\frac{ds}{dt}=-3\left(t^2+4\right)^{-2}(2t)$

$=\frac{-6t}{\left(t^2+4\right)^2}$

(c) $s=\frac{5}{\left(t^2-2\right)^3}=5\left(t^2-2\right)^{-3}$

$\frac{ds}{dt}=-15\left(t^2-2\right)^{-4}(2t)=\frac{-30t}{\left(t^2-2\right)^4}$

(d) $s=\frac{1}{\left(2t^2-t+2\right)^3}=\left(2t^2-t+2\right)^{-3}$

$\frac{ds}{dt}=-3\left(2t^2-t+2\right)^{-4}(4t-1)$

$\frac{ds}{dt}=\frac{-12t+3}{\left(2t^2-t+2\right)^4}$

7 (a) $y=x^2(x+1)^4$

$\frac{dy}{dt}=x^2(4)(x+1)^3(1)+(x+1)^4(2x)$

$=2x(x+1)^3\left[2x+(x+1)\right]$

$=2x(x+1)^3\left(3x+1\right)$

(b) $y=(2x+1)(x+3)^3$

$\frac{dy}{dx}=(2x+1)(3)(x+3)^2+(x+3)^3(2)$

$=(x+3)^2\left[6x+3+2(x+3)\right]$

$=(x+3)^2(8x+9)$

(c) $y=(x+2)^4(2x-3)^3$

$\frac{dy}{dx}=(x+2)^4(3)(2x-3)^2(2)+(2x-3)^3(4)(x+2)^3$

$\frac{dy}{dx}=6(x+2)^4(2x-3)^2+4(2x-3)^3(x+2)^3$

$\frac{dy}{dx}=2(x+2)^3(2x-3)^2\left[3(x+2)+2(2x-3)\right]$

$\frac{dy}{dx}=2(x+2)^3(2x-3)^2\left[7x\right]$

$\frac{dy}{dx}=14x(x+2)^3(2x-3)^2$

8 (a) $f(x)=x(3-x)^4$

$f'(x)=x(4)(3-x)^3(-1)+(3-x)^4$

$f'(x)=-4x(3-x)^3+(3-x)^4$

$f'(x)=(3-x)^3\left[-4x+3-x\right]$

$f'(x)=(3-x)^3\left(3-5x\right)$

(b) $f'(x)=0$

$(3-x)^3\left[3-5x\right]=0$

$x=3$ or $\frac{3}{5}$

9 (a) $y=\frac{3-x}{x^2+1}$

$\frac{dy}{dx}=\frac{\left(x^2+1\right)(-1)-(3-x)(2x)}{\left(x^2+1\right)^2}$

$\frac{dy}{dx}=\frac{\left(-x^2-1\right)-(6x+2x^2)}{\left(x^2+1\right)^2}$

$\frac{dy}{dx}=\frac{\left(-x^2-1\right)-6x-2x^2}{\left(x^2+1\right)^2}$

$\frac{dy}{dx}=\frac{-3x^2-6x-1}{\left(x^2+1\right)^2}$

(b) $y=\dfrac{(3x+1)^3}{2x+1}$

$\dfrac{dy}{dx}=\dfrac{(2x+1)(3)(3x+1)^2(3)-(3x+1)^3(2)}{(2x+1)^2}$

$\dfrac{dy}{dx}=\dfrac{9(2x+1)(3x+1)^2-2(3x+1)^3}{(2x+1)^2}$

$\dfrac{dy}{dx}=\dfrac{(3x+1)^2[18x+9-2(3x+1)]}{(2x+1)^2}$

$\dfrac{dy}{dx}=\dfrac{(3x+1)^2(12x+7)}{(2x+1)^2}$

(c) $y=\dfrac{2x-1}{(x^2+3)^2}$

$\dfrac{dy}{dx}=\dfrac{(x^2+3)^2(2)-(2x-1)(2)(x^2+3)(2x)}{(x^2+3)^4}$

$\dfrac{dy}{dx}=\dfrac{2(x^2+3)[x^2+3-2x(2x-1)]}{(x^2+3)^4}$

$\dfrac{dy}{dx}=\dfrac{2(-3x^2+2x+3)}{(x^2+3)^3}$

10 (a) $y=t^3-2t$

$\dfrac{dy}{dx}=3t^2-2$

$x=t^2+3t$

$\dfrac{dx}{dt}=2t+3$

$\dfrac{dy}{dx}=\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}=\dfrac{3t^2-2}{2t+3}$

(b) $y=(t-1)^2=t^2-2t+1$

$\dfrac{dy}{dt}=2t-2$

$x=t^2-1$

$\dfrac{dx}{dt}=2t$

$\dfrac{dy}{dx}=\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}=\dfrac{2t-2}{2t}=\dfrac{t-1}{t}$

11 (a) $y=3t^2$

$\dfrac{dy}{dt}=6t$

$x=2t+1$ $\qquad$ $x=2t+1$

$\dfrac{dx}{dt}=2$ $\qquad$ $t=\dfrac{x-1}{2}$

$\dfrac{dy}{dt}=\dfrac{\frac{dy}{dt}}{\frac{dx}{dt}}=\dfrac{6t}{2}=3t=3\left(\dfrac{x-1}{2}\right)$

(b) $y=\dfrac{1}{4}u^8=\dfrac{1}{4}(5x-2)^8$

$\dfrac{dy}{dx}=2(5x-2)^7(5)$

$=10(5x-2)^7$

UPSKILL 2.3

1 (a) $y=2x^3+4x^2-6x-3$

$\dfrac{dy}{dx}=6x^2+8x-6$

$\dfrac{d^2y}{dx^2}=12x+8$

(b) $y=\dfrac{2x+5}{x}=2+\dfrac{5}{x}$

$\dfrac{dy}{dx}=-\dfrac{5}{x^2}$

$\dfrac{d^2y}{dx^2}=\dfrac{10}{x^3}$

(c) $y=\dfrac{3x}{x+3}$

$\dfrac{dy}{dx}=\dfrac{(x+3)(3)-3x(1)}{(x+3)^2}$

$\dfrac{dy}{dx}=\dfrac{9}{(x+3)^2}=9(x+3)^{-2}$

$\dfrac{d^2y}{dx^2}=-18(x+3)^{-3}=-\dfrac{18}{(x+3)^3}$

(d) $y=(2x+1)(3x^2-2)=6x^3-4x+3x^2-2$

$\dfrac{dy}{dx}=18x^2-4+6x$

$\dfrac{d^2y}{dx^2}=36x+6$

(e) $y=(2x+1)^5$

$\frac{dy}{dx}=5(2x+1)^4(2)=10(2x+1)^4$

$\frac{d^2y}{dx^2}=40(2x+1)^3(2)=80(2x+1)^3$

2 (a) $f(x)=x^3-6x^2+9x+2$

$f'(x)=3x^2-12x+9$

$f''(x)=6x-12$

(b) $f(x)=x^2-\frac{2}{x}$

$f'(x)=2x+\frac{2}{x^2}$

$f''(x)=2-\frac{4}{x^3}$

(c) $f(x)=\frac{1}{3x+4}=(3x+4)^{-1}$

$f'(x)=-(3x+4)^{-2}(3)$

$f''(x)=6(3x+4)^{-3}(3)=\frac{18}{(3x+4)^3}$

(d) $f(x)=\frac{4x}{3x+1}$

$f'(x)=\frac{(3x+1)(4)-4x(3)}{(3x+1)^2}$

$=\frac{4}{(3x+1)^2}=4(3x+1)^{-2}$

$f''(x)=-8(3x+1)^{-3}(3)=-\frac{24}{(3x+1)^3}$

(e) $f(x)=\left(2x+\frac{1}{x}\right)^2=4x^2+4+x^{-2}$

$f'(x)=8x-2x^{-3}$

$f''(x)=8+6x^{-4}=8+\frac{6}{x^4}$

UPSKILL 2.4a

1 (a) $y=\left(x^2+3\right)^3$

$\frac{dy}{dx}=3(x^2+3)^2(2x)$

$\frac{dy}{dx}=6x\left(x^2+3\right)^2$

$m=6(-1)\left[(-1)^2+3\right]^2=-96$

(b) $y=\frac{x^2+3}{4x+1}$

$\frac{dy}{dx}=\frac{(4x+1)(2x)-(x^2+3)(4)}{(4x+1)^2}$

$\frac{dy}{dx}=\frac{8x^2+2x-4x^2-12}{(4x+1)^2}$

$\frac{dy}{dx}=\frac{4x^2+2x-12}{(4x+1)^2}$

$m=\frac{-12}{1^2}=-12$

(c) $y=\left(9-x^3\right)(1+x^2)=9+9x^2-x^3-x^5$

$\frac{dy}{dx}=18x-3x^2-5x^4$

$m=18(2)-3(2)^2-5(2)^4=-56$

(d) $y=\frac{x+1}{x^2}=x^{-1}+x^{-2}$

$\frac{dy}{dx}=-x^{-2}-2x^{-3}$

$\frac{dy}{dx}=-\frac{1}{x^2}-\frac{2}{x^3}$

$m=-\frac{1}{2^2}-\frac{2}{2^3}=-\frac{1}{4}-\frac{1}{4}=-\frac{1}{2}$

(e) $y=\frac{x-2}{x+1}$

$\frac{dy}{dx}=\frac{(x+1)-(x-2)}{(x+1)^2}=\frac{3}{(x+1)^2}$

$m=\frac{3}{(2+1)^2}=\frac{3}{9}=\frac{1}{3}$

2 $y=hx^2+kx$

The curve passes through the point $\left(\frac{1}{4},\frac{7}{8}\right)$.

$\frac{7}{8}=h\left(\frac{1}{4}\right)^2+k\left(\frac{1}{4}\right)$

$\frac{7}{8}=h\left(\frac{1}{16}\right)+k\left(\frac{1}{4}\right)$

$14=h+4k$ … (1)

$\frac{dy}{dx}=2hx+k$

$4=2h\left(\frac{1}{4}\right)+k$

$8 = h + 2k \quad \ldots (2)$

$(1) - (2): \quad 2k = 6$
$k = 3$

Substitute $k = 3$ into (2) :
$8 = h + 2(3)$
$h = 2$

3 $y = \dfrac{a}{x} + bx$

The curve passes through the point (2, 7).

$7 = \dfrac{a}{2} + 2b$

$14 = a + 4b \quad \ldots (1)$

$\dfrac{dy}{dx} = \dfrac{-a}{x^2} + b$

$\dfrac{5}{2} = \dfrac{-a}{2^2} + b$

$10 = -a + 4b \ldots (2)$

$(1) + (2): \quad 24 = 8b$
$b = 3$

Substitute $b = 3$ into (1) :
$14 = a + 4(3)$
$a = 2$

4 $y = fx + \dfrac{g}{x^2}$

The curve passes through the point (2, 5).

$5 = 2f + \dfrac{g}{2^2}$

$20 = 8f + g \quad \ldots (1)$

$\dfrac{dy}{dx} = f - \dfrac{2g}{x^3}$

$1 = f - \dfrac{2g}{2^3}$

$1 = f - \dfrac{g}{4}$

$4 = 4f - g \quad \ldots (2)$

$(1) + (2): \; 24 = 12f$
$f = 2$

Substitute $f = 2$ into (2) :
$4 = 4(2) - g$
$g = 4$

UPSKILL 2.4b

1 (a) $y = 3x^2 - 2x - 5$

$m = \dfrac{dy}{dx} = 6x - 2$

$m = 6(2) - 2 = 10$

The equation of the tangent is
$y - 3 = 10(x - 2)$
$y = 10x - 20 + 3$
$y = 10x - 17$

(b) $y = x + \dfrac{3}{x^2}$

$m = \dfrac{dy}{dx} = 1 - \dfrac{6}{x^3}$

$m = 1 - \dfrac{6}{1^3} = -5$

The equation of the tangent is
$y - 4 = -5(x - 1)$
$y = -5x + 5 + 4$
$y = -5x + 9$

2 When $x = 4$, $y = x^3 - 8x^2 + 14x$

$y = 4^3 - 8(4)^2 + 14(4) = -8$

$y = x^3 - 8x^2 + 14x$

$m = \dfrac{dy}{dx} = 3x^2 - 16x + 14$

$m = 3(4)^2 - 16(4) + 14 = -2$

The equation of the tangent is
$y - (-8) = -2(x - 4)$
$y + 8 = -2x + 8$
$y = -2x$

3 $y = \dfrac{3 + x}{3 - 2x}$

$4 = \dfrac{3 + x}{3 - 2x}$

$12 - 8x = 3 + x$
$9 = 9x$
$x = 1$

$m = \dfrac{dy}{dx} = \dfrac{(3 - 2x)(1) - (3 + x)(-2)}{(3 - 2x)^2}$

$m = \dfrac{3 - 2x + 2(3 + x)}{(3 - 2x)^2}$

$m = \dfrac{3 - 2x + 6 + 2x}{(3 - 2x)^2}$

$m = \frac{9}{(3-2x)^2}$

When $x = 1$, $m = \frac{9}{(3-2)^1} = 9$

(1, 4), $m = 9$

The equation of the tangent is

$y - 4 = 9(x-1)$

$y - 4 = 9x - 9$

$y = 9x - 5$

4 (a) $y = \frac{1}{3}x^3 - \frac{1}{2}x^2 - 3x - 9$

$m = \frac{dy}{dx} = x^2 - x - 3$

When $x = 6$, $m = 6^2 - 6 - 3 = 27$

Gradient of normal = $-\frac{1}{27}$

The equation of the normal is

$y - 27 = -\frac{1}{27}(x-6)$

$27y - 729 = -x + 6$

$27y = -x + 735$

(b) $y = 2x + \frac{8}{x}$

$m = \frac{dy}{dx} = 2 - \frac{8}{x^2}$

$m = 2 - \frac{8}{(-1)^2} = -6$

Gradient of normal = $\frac{1}{6}$

The equation of the normal is

$y - (-10) = \frac{1}{6}(x - (-1))$

$6y + 60 = x + 1$

$6y = x - 59$

5 $y = \frac{x^2+2}{x} = x + \frac{2}{x}$

$\frac{dy}{dx} = 1 - \frac{2}{x^2} = \frac{1}{2}$

$\frac{2}{x^2} = \frac{1}{2}$

$x^2 = 4$

$x = \pm 2$

$y = \frac{(\pm 2)^2 + 2}{\pm 2}$

$y = \frac{4+2}{2}$ or $y = \frac{4+2}{-2}$

$y = 3$ or -3

The required points are (2, 3) or (–2, –3).

6 $y = 2x^2 - x + 5$

$m = \frac{dy}{dx} = 4x - 1$

$m = 4(1) - 1 = 3$

The equation of the tangent is

$y - 6 = 3(x-1)$

$y = 3x + 3$

Gradient of perpendicular line $= -\frac{1}{3}$.

$4x - 1 = -\frac{1}{3}$

$4x = \frac{2}{3}$

$x = \frac{1}{6}$

7 The gradient of the line $10y = x + 5$ is $\frac{1}{10}$.

Thus, the gradient of the tangent is -10.

$y = 3x^2 + hx + k$

$\frac{dy}{dx} = 6x + h$

$-10 = 6(-2) + h$

$h = 2$

The curve passes through the point $(-2, 9)$.

$y = 3x^2 + 2x + k$

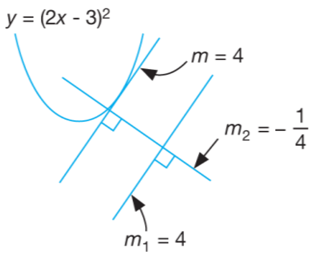
$9 = 3(-2)^2 + 2(-2) + k$

$9 = 12 - 4 + k$

$k = 1$

8 The gradient of the line $y = 4x - 2$ is 4.

Gradient of normal is $-\frac{1}{4}$.



Thus, the gradient of the tangent is 4.

$y = (2x-3)^2$

$$\frac{dy}{dx} = 2(2x-3)(2) = 4$$
$$8x-12=4$$
$$x=2$$

When $x = 2$, $y=(2\times 2-3)^2=1$

The required point is (1, 2).
The equation of the normal is

$$y-1=-\frac{1}{4}(x-2)$$
$$4y-4=-x+2$$
$$4y=-x+6$$

9 (a) Gradient of tangent = $\frac{18-2}{7+1}$ = 2

$$m=\frac{dy}{dx}=2$$
$$2x-4=2$$
$$x=3$$

When $x = 3$, $y = 3^2-4(3)-3=-6$

The coordinates of point R are (3, –6).

(b) Gradient of normal = $-\frac{1}{2}$

The equation of the normal is

$$y-(-6)=-\frac{1}{2}(x-3)$$
$$2y+12=-x+3$$
$$2y=-x-9$$

UPSKILL 2.4c

1 (a) $y=3x^2-x^3$

$$\frac{dy}{dx}=6x-3x^2$$
$$\frac{d^2y}{dx^2}=6-6x$$

At the turning points,

$$\frac{dy}{dx}=0$$
$$6x-3x^2=0$$
$$3x(2-x)=0$$
$$x=0 \text{ or } x=2$$

When $x = 0$,

$$y=3(0)^2-(0)^3=0$$

Thus, (0, 0) is a turning point.

$$\frac{d^2y}{dx^2}=6-6(0)=6 \text{ (Positive)}$$

Hence, the minimum point is (0, 0).

When $x = 2$,

$$y=3(2)^2-(2)^3=4$$

Thus, (2, 4) is a turning point.

$$\frac{d^2y}{dx^2}=6-6(2)=-6 \text{ (Negative)}$$

Hence, (2, 4) is a maximum point.

(b) $y=4x+\frac{9}{x}$

$$\frac{dy}{dx}=4-\frac{9}{x^2}$$
$$\frac{d^2y}{dx^2}=\frac{18}{x^3}$$

At the turning points,

$$\frac{dy}{dx}=0$$
$$4-\frac{9}{x^2}=0$$
$$\frac{9}{x^2}=4$$
$$x^2=\frac{9}{4}$$
$$x=\pm\frac{3}{2}$$
$$\frac{d^2y}{dx^2}=\frac{18}{x^3}$$

When $x=\frac{3}{2}$,

$$y=4\left(\frac{3}{2}\right)+\frac{9}{\left(\frac{3}{2}\right)}=12$$

$\left(\frac{3}{2},12\right)$ is a turning point.

$$\frac{d^2y}{dx^2}=\frac{18}{\left(\frac{3}{2}\right)^3}=\frac{16}{3} \text{ (Positive)}$$

Hence, $\left(\frac{3}{2},12\right)$ is a minimum point.

When $x=-\frac{3}{2}$,

$$y=4\left(-\frac{3}{2}\right)+\frac{9}{\left(-\frac{3}{2}\right)}=-12$$
$$\frac{d^2y}{dx^2}=\frac{18}{\left(-\frac{3}{2}\right)^3}=-\frac{16}{3} \text{ (Negative)}$$

$\left(-\frac{3}{2},-12\right)$ is a maximum point.

(c) $y = 4 - x^2 - \frac{16}{x^2}$

$\frac{dy}{dx} = -2x + \frac{32}{x^3}$

$\frac{d^2y}{dx^2} = -2 - \frac{96}{x^4}$

At the turning points,

$\frac{dy}{dx} = 0$

$-2x + \frac{32}{x^3} = 0$

$\frac{32}{x^3} = 2x$

$x^4 = 16$

$x = \pm 2$

When $x = 2$,

$y = 4 - 2^2 - \frac{16}{2^2}$

$y = -4$

Hence, (2, –4) is a turning point.

At the turning points,

$\frac{dy}{dx} = 0$

$-2x + \frac{32}{x^3} = 0$

$\frac{32}{x^3} = 2x$

$x^4 = 16$

$x = \pm 2$

When $x = 2$,

$y = 4 - 2^2 - \frac{16}{2^2} = -4$

(2, –4) is a turning point.

$\frac{d^2y}{dx^2} = -2 - \frac{96}{2^4} = -8$ (Negative)

Hence, (2, –4) is a maximum point.

When $x = -2$,

$y = 4 - (-2)^2 - \frac{16}{(-2)^2} = -4$

(–2, –4) is a turning point.

$\frac{d^2y}{dx^2} = -2 - \frac{96}{(-2)^4} = -8$ (Negative)

Hence, (–2, –4) is a maximum point.

(d) $y = x(x-3)^2$

$y = x(x^2 - 6x + 9)$

$y = x^3 - 6x^2 + 9x$

$\frac{dy}{dx} = 3x^2 - 12x + 9$

$\frac{d^2y}{dx^2} = 6x - 12$

At the turning points,

$\frac{dy}{dx} = 0$

$3x^2 - 12x + 9 = 0$

$x^2 - 4x + 3 = 0$

$(x-3)(x-1) = 0$

$x = 3$ or $x = 1$

When $x = 3$,

$y = 3(3-3)^2 = 0$

Thus, (3, 0) is a turning point.

$\frac{d^2y}{dx^2} = 6(3) - 12 = 6$ (Positive)

Hence, (3, 0) is a minimum point.

When $x = 1$,

$y = (1)(1-3)^2 = 4$

Hence, (1, 4) is a turning point.

$\frac{d^2y}{dx^2} = 6(1) - 12 = -6$ (Negative)

Hence, (1, 4) is a maximum point.

(e) $y = x^3 + 3x^2 - 9x - 1$

$\frac{dy}{dx} = 3x^2 + 6x - 9$

$\frac{d^2y}{dx^2} = 6x + 6$

At the turning points,

$\frac{dy}{dx} = 0$

$3x^2 + 6x - 9 = 0$

$x^2 + 2x - 3 = 0$

$(x+3)(x-1) = 0$

$x = -3$ or $x = 1$

When $x = -3$,

$y = (-3)^3 + 3(-3)^2 - 9(-3) - 1 = 26$

Thus, (–3, 26) is a turning point.

$\frac{d^2y}{dx^2} = 6(-3) + 6 = -12$ (Negative)

Hence, $(-3, 26)$ is a maximum point.

When $x=1$, $y=1+3-9-1=-6$
Thus, (1, –6) is a turning point.

$$\frac{d^2y}{dx^2}=6+6=12 \text{ (Positive)}$$

Hence, (1, –6) is a minimum point.

2 (a) $y = hx^2 + \frac{k}{x^2}$

The curve passes through the point (1, 17).
$17 = h + k \quad \ldots (1)$

$$\frac{dy}{dx} = 2hx - \frac{2k}{x^3}$$
$$-30 = 2h - 2k$$
$$-15 = h - k \quad \ldots (2)$$

$(1) + (2) : 2h = 2$
$h = 1$

From (1) :
$17 = 1 + k$
$k = 16$

(b) $y = x^2 + \frac{16}{x^2}$

$$\frac{dy}{dx} = 2x - \frac{32}{x^3}$$
$$\frac{d^2y}{dx^2} = 2 + \frac{96}{x^4}$$

At the turning points,

$$\frac{dy}{dx} = 0$$
$$2x - \frac{32}{x^3} = 0$$
$$2x = \frac{32}{x^3}$$
$$x^4 = 16$$
$$x = \pm 2$$

When $x = 2$,

$$y = 2^2 + \frac{16}{2^2} = 8$$

Thus, (2, 8) is a turning point.

$$\frac{d^2y}{dx^2} = 2 + \frac{96}{2^4} = 8 \text{ (Positive)}$$

Hence, (2, 8) is a minimum point.

When $x = -2$,

$$y = (-2)^2 + \frac{16}{(-2)^2} = 8$$

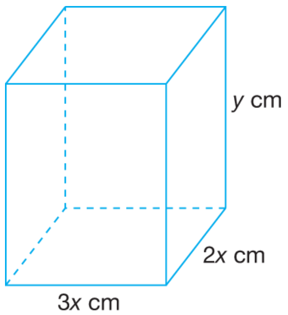
Thus, (–2, 8) is a turning point.

$$\frac{d^2y}{dx^2} = 2 + \frac{96}{(-2)^4} = 8 \text{ (Positive)}$$

Hence, (–2, 8) is a minimum point.

UPSKILL 2.4d

1



Let the height of the cuboid = y cm

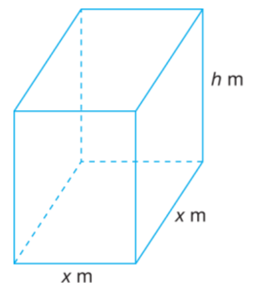
$$4(3x) + 4(2x) + 4y = 200$$
$$3x + 2x + y = 50$$
$$y = 50 - 5x$$

$$V = (3x)(2x)(y)$$
$$V = 6x^2y$$
$$V = 6x^2(50 - 5x)$$
$$V = 30x^2(10 - x) \text{ [Shown]}$$
$$V = 300x^2 - 30x^3$$

$$\frac{dV}{dx} = 600x - 90x^2 = 0$$
$$30x(20 - 3x) = 0$$
$$x = \frac{20}{3}$$

$$V_{maximum} = 300\left(\frac{20}{3}\right)^2 - 30\left(\frac{20}{3}\right)^3$$
$$= \frac{40\ 000}{9} \text{ cm}^3$$

2



$V = x^2h = 8$

$h = \dfrac{8}{x^2}$

$A = 2x^2 + 4hx$

$A = 2x^2 + 4x\left(\dfrac{8}{x^2}\right)$

$A = 2x^2 + \dfrac{32}{x}$

$\dfrac{dA}{dx} = 4x - \dfrac{32}{x^2}$

When $\dfrac{dA}{dx} = 4x - \dfrac{32}{x^2} = 0$,

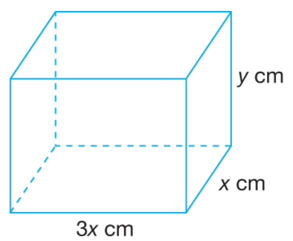
$\dfrac{32}{x^2} = 4x$

$x^3 = 8$

$x = 2$

$h = \dfrac{8}{x^2} = \dfrac{8}{2^2} = 2$

3



$A = 2(3x^2) + 2xy + 2(3xy) = 1274\text{ cm}^2$

$6x^2 + 8xy = 1274$

$y = \dfrac{1274 - 6x^2}{8x}$

$P = 4(3x) + 4x + 4y$

$P = 16x + 4y$

$P = 16x + 4\left(\dfrac{1274 - 6x^2}{8x}\right)$

$P = 16x + \left(\dfrac{1274 - 6x^2}{2x}\right)$

$P = 16x + \left(\dfrac{637 - 3x^2}{x}\right)$

$P = 16x + \left(\dfrac{637}{x} - 3x\right)$

$P = 13x + \dfrac{637}{x}$

$\dfrac{dP}{dx} = 13 - \dfrac{637}{x^2} = 0$

$\dfrac{637}{x^2} = 13$

$x^2 = 49$

$x = 7$

$P_{\text{minimum}} = 13(7) + \dfrac{637}{7} = 182\text{ cm}$

4 (a) Let $AB = DC = y$ cm

$P = 2y + 2\pi r = 120$

$y + \pi r = 60$

$y = 60 - \pi r$

$L = 2ry - \pi r^2$

$L = 2r(60 - \pi r) - \pi r^2$

$L = 120r - 3\pi r^2$ [Shown]

(b) $\dfrac{dL}{dr} = 120 - 6\pi r = 0$

$6\pi r = 120$

$r = \dfrac{120}{6\pi}$

$r = \dfrac{20}{\pi}$

$\dfrac{d^2L}{dr^2} = -6\pi$ [< 0]

Thus, the value of L is a maximum.

5 (a) $V = 96\text{ cm}^3$

$\dfrac{1}{2}(4x)(3x)(y) = 96$

$6x^2y = 96$

$y = \dfrac{96}{6x^2}$

$y = \dfrac{16}{x^2}$

$A = (4x)(3x) + 3xy + 4xy + 5xy$

$A = 12x^2 + 12xy$

$A = 12x^2 + 12x\left(\frac{16}{x^2}\right)$

$A = 12x^2 + \frac{192}{x}$ [Shown]

(b) $\frac{dA}{dx} = 24x - \frac{192}{x^2} = 0$

$\frac{192}{x^2} = 24x$

$x^3 = 8$

$x = 2$

$h = 3x = 3(2) = 6$ cm

6 (a) $V = \pi r^2 h = 192\pi$

$h = \frac{192}{r^2}$

$A = \pi r^2 + 2\pi rh + 2\pi r^2$

$A = 3\pi r^2 + 2\pi r\left(\frac{192}{r^2}\right)$

$A = 3\pi r^2 + 2\pi\left(\frac{192}{r}\right)$

$A = 3\pi r^2 + \frac{384\pi}{r}$ [Shown]

(b) $\frac{dA}{dr} = 6\pi r - \frac{384\pi}{r^2} = 0$

$6\pi r = \frac{384\pi}{r^2}$

$r^3 = 64$

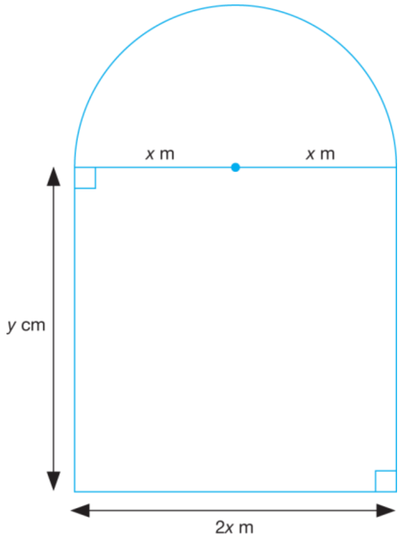
$r = 4$

$A_{maximum} = 3\pi(4)^2 + \pi\left(\frac{384}{4}\right)$

$= 48\pi + 96\pi$

$= 144\pi$ cm^2

7



(a) Perimeter = 6 m

$2y + 2x + \pi x = 6$

$2y = 6 - 2x - \pi x$

$y = \frac{6 - 2x - \pi x}{2}$

A = Area of rectangle + Area of semicircle

$= 2xy + \frac{1}{2}\pi x^2$

$= 2x\left(\frac{6 - 2x - \pi x}{2}\right) + \frac{1}{2}\pi x^2$

$= 6x - 2x^2 - \pi x^2 + \frac{1}{2}\pi x^2$

$= 6x - 2x^2 - \frac{1}{2}\pi x^2$

(b) $\frac{dA}{dx} = 6 - 4x - \pi x$

When A has a stationary value,

$\frac{dA}{dx} = 0$

$6 - 4x - \pi x = 0$

$4x + \pi x = 6$

$(4 + \pi)x = 6$

$x = \frac{6}{4 + \pi}$

$$\frac{d^2A}{dx^2} = -4 - \pi \text{ (Negative)}$$

Hence, A is a maximum.

Hence, when the surface area of the window is a maximum, the width of the window

$= 2x$

$= 2\left(\frac{6}{4 + 3.142}\right)$

$= 1.680$ m

UPSKILL 2.4e

1 $A = \pi r^2$

$\frac{dA}{dr} = 2\pi r$

$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt} = 2\pi(4) \times 0.5 = 4\pi \text{ cm}^2\text{ s}^{-1}$

2 (a) $V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt} = 4\pi r^2 \times 0.5$

$= 4\pi(4)^2 \times 0.5$

$= 32\pi \text{ cm}^3\text{ s}^{-1}$

(b) $A = 4\pi r^2$

$\frac{dA}{dr} = 8\pi r$

$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt} = 8\pi(4) \times 0.5 = 16\pi \text{ cm}^2\text{ s}^{-1}$

3 $V = \pi r^2 h = \pi(30)^2 h = 900\pi h$

$\frac{dV}{dh} = 900\pi$

$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$

$= \frac{1}{900\pi} \times 90$

$= \frac{1}{10\pi} \text{ cm s}^{-1}$

4 $V = 6\pi h^2 - \frac{1}{3}\pi h^3$

$\frac{dV}{dh} = 12\pi h - \pi h^2$

$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$

$= \frac{1}{12\pi h - \pi h^2} \times 3$

$= \frac{1}{12\pi(2) - \pi(2)^2} \times 3$

$= \frac{1}{20\pi} \times 3$

$= \frac{3}{20\pi} \text{ cm s}^{-1}$

5 Let the area of $\Delta ABC = L$ cm^2

$L = \frac{1}{2}(x)(3x)\sin 150^{\circ}$

$L = \frac{3}{2}x^2\left(\frac{1}{2}\right)$

$L = \frac{3}{4}x^2$

$\frac{dL}{dx} = \frac{3}{2}x$

$\frac{dL}{dt} = \frac{dL}{dx} \times \frac{dx}{dt}$

$= \frac{3}{2}x \times (0.5)$

$= \frac{3}{4}x$

$= \frac{3}{4} \times 4$

$= 3 \text{ cm}^2\text{ s}^{-1}$

6 $V = \frac{4}{3}\pi r^3$

$\frac{dV}{dr} = 4\pi r^2$

$\frac{dr}{dt} = \frac{dr}{dV} \times \frac{dV}{dt}$

$= \frac{1}{4\pi r^2} \times 10\pi$

$= \frac{1}{4\pi(4)^2} \times 10\pi$

$= \frac{5}{32} \text{ cm s}^{-1}$

7 (a) $V = x^3$

$\frac{dV}{dx} = 3x^2$

$$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt} = 3(5)^2 \times (-2) = -150 \text{ cm}^3 \text{ s}^{-1}$$

(b) $A = 6x^2$

$$\frac{dA}{dx} = 12x$$

$$\frac{dA}{dt} = \frac{dA}{dx} \times \frac{dx}{dt} = 12(5) \times (-2) = -120 \text{ cm}^2 \text{ s}^{-1}$$

8 $\frac{h}{16} = \frac{r}{4}$

$$r = \frac{4}{16}h = \frac{h}{4}$$

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{4}\right)^2 h = \frac{1}{48}\pi h^3$$

$$\frac{dV}{dh} = \frac{1}{16}\pi h^2$$

$$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$$

$$= \frac{16}{\pi(8)^2} \times 2$$

$$= \frac{1}{2\pi} \text{ cm s}^{-1}$$

UPSKILL 2.4f

1 $y = 2x^3 - 8x^2 + 11$

$$\frac{dy}{dx} = 6x^2 - 16x$$

$$\delta y \approx \frac{dy}{dx} \times \delta x$$

$$\approx (6x^2 - 16x) \times 0.02$$

$$\approx [6(2)^2 - 16(2)] \times 0.02$$

$$\approx -0.16$$

The value of y decreases.

2 $y = 4x - \frac{6}{x}$

$$\frac{dy}{dx} = 4 + \frac{6}{x^2}$$

$$\delta y \approx \frac{dy}{dx} \times \delta x \approx \left(4 + \frac{6}{x^2}\right) \times (2 + k - 2)$$

$$\delta y \approx \left(4 + \frac{6}{2^2}\right) \times k \approx \frac{11}{2}k$$

3 $A = \pi r^2$

$$\frac{dA}{dr} = 2\pi r$$

$$\delta A \approx \frac{dA}{dr} \times \delta r \approx 2\pi(5) \times (0.03) \approx 0.3\pi \text{ cm}^2$$

4 $V = \frac{1}{3}\pi r^2 h$

$$V = \frac{1}{3}\pi r^2 (5)$$

$$V = \frac{5}{3}\pi r^2$$

$$\frac{dV}{dr} = \frac{10}{3}\pi r$$

$$\delta V \approx \frac{dV}{dr} \times \delta r$$

$$\approx \frac{10}{3}\pi r \times (3.15 - 3)$$

$$\approx \frac{10}{3}\pi(3) \times (3.15 - 3)$$

$$\approx 1.5\pi \text{ cm}^3$$

5 $V = \pi r^2 h$

$$V = \pi(4)^2 h$$

$$V = 16\pi h$$

$$\delta V \approx \frac{dV}{dh} \times \delta h$$

$$\approx 16\pi \times 0.25$$

$$\approx 4\pi \text{ cm}^3$$

6 $V = \frac{4}{3}\pi r^3$

$$\frac{dV}{dr} = 4\pi r^2$$

$$\delta V \approx \frac{dV}{dr} \times \delta r$$

$$\approx 4\pi r^2 \times (-0.2)$$

$$\approx 4\pi(10)^2 \times (-0.2)$$

$$\approx 80\pi \text{ cm}^3$$

7 $A = 6x^2$

$$\frac{dA}{dx} = 12x$$

$$\delta A \approx \frac{dA}{dx} \times \delta x \approx 12x(-0.1) \approx -1.2(6)$$

$$\approx -7.2 \text{ cm}^2$$

8 $y = \dfrac{5}{x^3}$

$\dfrac{dy}{dx} = -\dfrac{15}{x^4}$

When $x = 4$, $\dfrac{dy}{dx} = -\dfrac{15}{4^4} = -\dfrac{15}{256}$

(a) $y_{\text{new}} = y_{\text{original}} + \dfrac{dy}{dx} \times \delta x$

$$\frac{5}{4.02^3} = \frac{5}{4^3} + \left(-\frac{15}{x^4}\right)(0.02)$$
$$= \frac{5}{64} + \left(-\frac{15}{4^4}\right)(0.02)$$
$= 0.0770$

(b) $y_{\text{new}} = y_{\text{original}} + \dfrac{dy}{dx} \times \delta x$

$$\frac{5}{3.99^3} = \frac{5}{4^3} + \left(-\frac{15}{x^4}\right)(-0.01)$$
$$= \frac{5}{64} + \left(-\frac{15}{4^4}\right)(-0.01)$$
$= 0.0787$

Summative Practice 2

1 $\lim_{x \to -4} \left(\dfrac{x^2 - 16}{x^2 + 4x}\right)$

$= \lim_{x \to -4} \left[\dfrac{(x+4)(x-4)}{x(x+4)}\right]$

$= \lim_{x \to -4} \left[\dfrac{(x-4)}{x}\right]$

$= \dfrac{-4-4}{-4}$

$= 2$

2 $y = x + \dfrac{1}{x}$ … (1)

$y + \delta y = x + \delta x + \dfrac{1}{x + \delta x}$ … (2)

(2) – (1) :

$\delta y = \delta x + \dfrac{1}{x + \delta x} - \dfrac{1}{x}$

$\delta y = \delta x + \dfrac{x - (x + \delta x)}{x(x + \delta x)}$

$\delta y = \delta x + \dfrac{-\delta x}{x(x + \delta x)}$

$\dfrac{\delta y}{\delta x} = 1 + \dfrac{-1}{x(x + \delta x)}$

$\dfrac{dy}{dx} = \lim_{\delta x \to 0} \dfrac{\delta y}{\delta x}$

$\dfrac{dy}{dx} = \lim_{\delta x \to 0} \left[1 + \dfrac{-1}{x(x + \delta x)}\right]$

$\dfrac{dy}{dx} = 1 + \dfrac{-1}{x(x+0)}$

$\dfrac{dy}{dx} = 1 - \dfrac{1}{x^2}$

3 $y = x^2(1-3x)^3$

$\dfrac{dy}{dx} = x^2(3)(1-3x)^2(-3) + (1-3x)^3(2x)$

$= x(1-3x)^2[-9x + 2(1-3x)]$

$= x(1-3x)^2[-9x + 2 - 6x]$

$= x(1-3x)^2(2 - 15x)$

4 $y = \dfrac{2}{t}$

$\dfrac{dy}{dt} = -\dfrac{2}{t^2}$

$x = 2t - 1$

$\dfrac{dx}{dt} = 2$

$t = \dfrac{x+1}{2}$

$\dfrac{dy}{dx} = \dfrac{\frac{dy}{dt}}{\frac{dx}{dt}} = \dfrac{-\frac{2}{t^2}}{2} = -\dfrac{1}{t^2}$

$\dfrac{dy}{dx} = -\dfrac{1}{\left(\frac{x+1}{2}\right)^2} = -\dfrac{4}{(x+1)^2}$

5 $y = (3-x)^2(2x-3)^4$

$\dfrac{dy}{dx} = (3-x)^2(4)(2x-3)^3(2) +$

$(2x-3)^4(2)(3-x)(-1)$

$\dfrac{dy}{dx} = 2(3-x)(2x-3)^3[4(3-x) - (2x-3)]$

$\dfrac{dy}{dx} = 2(3-x)(2x-3)^3(15 - 6x)$

$\dfrac{dy}{dx} = 2(3-x)(2x-3)^3(3)(5 - 2x)$

$\dfrac{dy}{dx} = 6(3-x)(2x-3)^3(5 - 2x)$

6 $y = \frac{1}{3}u^6$

$y = \frac{1}{3}(3x-6)^6$

$\frac{dy}{dx} = \frac{6}{3}(3x-6)^5(3)$

$= 6(3x-6)^5$

7 $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{14t^3}{2} = 7t^3$

From $x = 2t+3$, $t = \frac{x-3}{2}$

Hence, $\frac{dy}{dx} = 7\left(\frac{x-3}{2}\right)^3$

$= \frac{7}{8}(x-3)^3$

8 $y = \frac{3x}{2x+5}$

$\frac{dy}{dx} = \frac{(2x+5)(3)-(3x)(2)}{(2x+5)^2}$

$= \frac{15}{(2x+5)^2}$

$\frac{dy}{dx} = 15(2x+5)^{-2}$

$\frac{d^2y}{dx^2} = -30(2x+5)^{-3}(2)$

$= \frac{-60}{(2x+5)^3}$

9 $y = px^2 + \frac{q}{x}$

The curve passes through the point (2, 3).

$3 = p(2)^2 + \frac{q}{2}$

$6 = 8p + q$ … (1)

$\frac{dy}{dx} = 2px - \frac{q}{x^2}$

$\frac{3}{2} = 2p(2) - \frac{q}{2^2}$

$6 = 16p - q$ … (2)

(1) + (2) :

$24p = 12$

$p = \frac{1}{2}$

From (1) :

$6 = 8\left(\frac{1}{2}\right) + q$

$q = 2$

10 If the gradient of normal is $-\frac{1}{9}$, hence the gradient of the tangent is 9.

$y = (3x-2)^3$

$\frac{dy}{dx} = 3(3x-2)^2(3)$

$\frac{dy}{dx} = 9(3x-2)^2$

$9 = 9(3-2x)^2$

$(3x-2)^2 = 1$

$9x^2 - 12x + 4 = 1$

$9x^2 - 12x + 3 = 0$

$3x^2 - 4x + 1 = 0$

$(x-1)(3x-1) = 0$

$x = 1$ or $x = \frac{1}{3}$ $y = (3x-2)^3$

When x = 1, $y = (3-2)^3 = 1$

When $x = \frac{1}{3}$, $y = \left(3\times\frac{1}{3} - 2\right)^3 = -1$

Hence, the required points are (1, 1) and $\left(\frac{1}{3}, -1\right)$.

11 (a) The equation of the normal is

$4y + x = p$

$4y = -x + p$

$y = -\frac{1}{4}x + \frac{p}{4}$

Gradient of normal = $-\frac{1}{4}$

Gradient of tangent = 4

$y = (2x-3)^2 - 4$

$\frac{dy}{dx} = 2(2x-3)(2)$

$\frac{dy}{dx} = 8x - 12$

$4 = 8x - 12$

$8x = 16$

$x = 2$

When $x = 2$,

$y = (2 \times 2 - 3)^2 - 4 = -3$

Hence, the coordinates of point Q are $(2, -3)$.

$4y + x = p$

At $(2, -3)$,

$4(-3) + 2 = p$

$p = -10$

(b) The equation of the tangent is

$y - (-3) = 4(x - 2)$

$y + 3 = 4x - 8$

$y = 4x - 11$

12 $y = \dfrac{2x - 6}{x + 3}$

$\dfrac{dy}{dx} = \dfrac{(x+3)(2) - (2x-6)(1)}{(x+3)^2}$

$\dfrac{dy}{dx} = \dfrac{2x + 6 - 2x + 6}{(x+3)^2}$

$\dfrac{dy}{dx} = \dfrac{12}{(x+3)^2}$

At point P (x-axis), $y = 0$

$y = \dfrac{2x - 6}{x + 3}$

$0 = \dfrac{2x - 6}{x + 3}$

$2x - 6 = 0$

$x = 3$

$\dfrac{dy}{dx} = \dfrac{12}{(3+3)^2}$

$m = \dfrac{12}{36} = \dfrac{1}{3}$

13 $y = \dfrac{1}{x^2} - \dfrac{1}{x^3}$

$\dfrac{dy}{dx} = -\dfrac{2}{x^3} + \dfrac{3}{x^4}$

$\dfrac{d^2y}{dx^2} = \dfrac{6}{x^4} - \dfrac{12}{x^5}$

$x^4\left(\dfrac{dy}{dx} + \dfrac{d^2y}{dx^2}\right) + x^2 y + 5 = 0$

$x^4\left(-\dfrac{2}{x^3} + \dfrac{3}{x^4} + \dfrac{6}{x^4} - \dfrac{12}{x^5}\right) + x^2\left(\dfrac{1}{x^2} - \dfrac{1}{x^3}\right) + 5 = 0$

$-2x + 9 - \dfrac{12}{x} + 1 - \dfrac{1}{x} + 5 = 0$

$-2x + 15 - \dfrac{13}{x} = 0$

$-2x^2 + 15x - 13 = 0$

$2x^2 - 15x + 13 = 0$

$(2x - 13)(x - 1) = 0$

$x = \dfrac{13}{2}$ or $x = 1$

14 $y = (x+1)^2(x-2)$

$y = \left(x^2 + 2x + 1\right)(x - 2)$

$y = x^3 + 2x^2 + x - 2x^2 - 4x - 2$

$y = x^3 - 3x - 2$

$\dfrac{dy}{dx} = 3x^2 - 3$

$\dfrac{d^2y}{dx^2} = 6x$

At the turning points,

$\dfrac{dy}{dx} = 0$

$3x^2 - 3 = 0$

$x^2 - 1 = 0$

$x^2 = 1$

$x = \pm 1$

When $x = 1$, $y = 1^3 - 3(1) - 2 = -4$

Thus, $(1, -4)$ is a turning point.

$\dfrac{d^2y}{dx^2} = 6(1) = 6$ (Positive)

Hence, $(1, -4)$ is a minimum point.

When $x = -1$, $y = (-1)^3 - 3(-1) - 2 = 0$

$\dfrac{d^2y}{dx^2} = 6(-1) = -6$ (Negative)

Hence, $(-1, 0)$ is a maximum point.

15 (a) $y = ax^3 + bx + c$

$\dfrac{dy}{dx} = 3ax^2 + b$

At the turning points,

$\dfrac{dy}{dx} = 0$

$3ax^2 + b = 0$

$3a(1)^2 + b = 0$

$3a + b = 0 \ \ldots (1)$

The curve passes through the point (1, 1).

$1 = a(1)^3 + b(1) + c$

$a + b + c = 1$ … (2)

The curve passes through the point (–1, 5).

$5 = a(-1)^3 + b(-1) + c$

$-a - b + c = 5$ … (3)

(2) – (3) : $2a + 2b = -4$

$a + b = -2$ … (4)

$3a + b = 0$ … (1)

(–) $a + b = -2$ … (4)

$2a = 2$

$a = 1$

From (1) :

$3(1) + b = 0$

$b = -3$

From (2) :

$1 - 3 + c = 1$

$c = 3$

(b) $y = x^3 - 3x + 3$

$\frac{dy}{dx} = 3x^2 - 3$

$\frac{d^2y}{dx^2} = 6x$

For (1, 1):

$\frac{d^2y}{dx^2} = 6(1) = 6$ (Positive)

Hence, (1, 1) is a minimum point.

For (–1, 5):

$\frac{d^2y}{dx^2} = 6(-1) = -6$ (Negative)

Hence, (–1, 5) is a maximum point.

16 (a) $y = \frac{x^3}{3} - \frac{x^2}{2} - 2x + \frac{1}{6}$

$\frac{dy}{dx} = \frac{3x^2}{3} - \frac{2x}{2} - 2$

$= x^2 - x - 2$

At the point $A(1, -2)$,

$m = \frac{dy}{dx} = 1^2 - 1 - 2 = -2$

(b) m (tangent) = –2

m (normal) = $\frac{1}{2}$

Equation of normal at the point $A(1, -2)$ is

$y - (-2) = \frac{1}{2}(x - 1)$

$2y + 4 = x - 1$

$2y = x - 5$

(c) At the turning points,

$\frac{dy}{dx} = 0$

$x^2 - x - 2 = 0$

$(x - 2)(x + 1) = 0$

$x = 2$ or –1

$x = -1$ is not accepted.

$\therefore x = 2$

When $x = 2$,

$y = \frac{2^3}{3} - \frac{2^2}{2} - 2(2) + \frac{1}{6} = -3\frac{1}{6}$

Hence, Q is point $\left(2, -3\frac{1}{6}\right)$.

$\frac{d^2y}{dx^2} = 2x - 1$

When $x = 2$,

$\frac{d^2y}{dx^2} = 2(2) - 1 = 3$ (Positive)

Hence, $\left(2, -3\frac{1}{6}\right)$ is a minimum point.

17 $y = 12 - x^3 - \frac{48}{x} = 12 - x^3 - 48x^{-1}$

$\frac{dy}{dx} = -3x^2 + 48x^{-2} = -3x^2 + \frac{48}{x^2}$

$\frac{d^2y}{dx^2} = -6x - 96x^{-3} = -6x - \frac{96}{x^3}$

At the turning points,

$\frac{dy}{dx} = 0$

$-3x^2 + \frac{48}{x^2} = 0$

$\frac{48}{x^2} = 3x^2$

$x^4 = \frac{48}{3}$

$x^4 = 16$

$x = \pm 2$

When $x = 2$,

$y = 12 - 2^3 - \frac{48}{2} = -20$

(2, –20) is a turning point.

$\frac{d^2y}{dx^2} = -6(2) - \frac{96}{2^3} = -24$ (Negative)

Hence, (2, –20) is a maximum point.

When $x = -2$,

$y = 12 - (-2)^3 - \frac{48}{(-2)} = 44$

(–2, 44) is a turning point.

$\frac{d^2y}{dx^2} = -6(-2) - \frac{96}{(-2)^3} = 24$ (Positive)

Hence, (–2, 44) is a minimum point.

18 (a) A = Area of rectangle – Area of triangle

$A = 60 \times 40 - \frac{1}{2}(3x)(40 - x)$

$A = 2\ 400 - 60x + \frac{3}{2}x^2$ [Shown]

(b) $\frac{dA}{dx} = -60 + 3x$

When A has a stationary value,

$\frac{dA}{dx} = 0$

$-60 + 3x = 0$

$x = 20$

$\frac{d^2A}{dx^2} = 3$ (Positive)

Hence, the minimum value of A

$= 2\ 400 - 60(20) + \frac{3}{2}(20)^2$

$= 1\ 800\text{ cm}^2$

19 (a) $PR^2 = x^2 + 10^2$

$PR = \sqrt{x^2 + 100}$

Radius of circle

$= \frac{\sqrt{x^2 + 100}}{2}$

A = Area of circle – Area of triangle

$A = \pi\left(\frac{\sqrt{x^2 + 100}}{2}\right)^2 - \frac{1}{2}x(10)$

$A = \pi\left(\frac{x^2 + 100}{4}\right) - 5x$

$A = 25\pi + \frac{\pi}{4}x^2 - 5x$ [Shown]

(b) When A has a stationary value,

$\frac{dA}{dx} = 0$

$\frac{1}{2}\pi x - 5 = 0$

$\pi x = 10$

$x = \frac{10}{\pi}$

$\frac{d^2A}{dx^2} = \frac{1}{2}\pi$ (Positive)

Hence, the minimum value of A

$= 25\pi + \frac{\pi}{4}\left(\frac{10}{\pi}\right)^2 - 5\left(\frac{10}{\pi}\right)$

$= 25\pi + \frac{25}{\pi} - \frac{50}{\pi}$

$= \left(25\pi - \frac{25}{\pi}\right)\text{cm}^2$

20 (a) Perimeter of the rectangle

$2x + 2y = 72$

$x + y = 36$

$y = 36 - x$

$V = \pi r^2 h$

$V = \pi\left(\frac{x}{2}\right)^2(y)$

$V = \pi\left(\frac{x^2}{4}\right)(36 - x)$

$V = \frac{\pi x^2}{4}(36 - x)$ [Shown]

$V = 9\pi x^2 - \frac{\pi}{4}x^3$

$\frac{dV}{dx} = 18\pi x - \frac{\pi}{4}(3x^2)$

$\frac{dV}{dx} = 18\pi x - \frac{3\pi}{4}x^2$

When V has a stationary value

$\frac{dV}{dx} = 0$

$18\pi x - \frac{3\pi}{4}x^2 = 0$

$18\pi x = \frac{3\pi}{4}x^2$

$$6x = \frac{1}{4}x^2$$
$$24x = x^2$$
$$x^2 - 24x = 0$$
$$x(x-24) = 0$$
$x = 0$ or $x = 24$

$x = 0$ is not accepted.

$\therefore x = 24$

$$\frac{d^2V}{dx^2} = 18\pi - \frac{3\pi}{2}x$$

When x = 24, $\frac{d^2V}{dx^2} = 18\pi - \frac{3\pi}{2}(24)$

$= -18\pi$ (Negative)

Hence, the maximum value of V

$= 9\pi(24)^2 - \frac{\pi}{4}(24)^3$

$= 5\ 184\pi - 3\ 456\pi$

$= 1\ 782\pi$ cm^3

(b) $A = 2\pi rh$

$$A = 2\pi\left(\frac{x}{2}\right)(y)$$
$$A = \pi x(36 - x)$$
$$A = 36\pi x - \pi x^2$$

$$\frac{dA}{dx} = 36\pi - 2\pi x$$

When A has a stationary value,

$$\frac{dA}{dx} = 0$$
$$36\pi - 2\pi x = 0$$
$$18 - x = 0$$
$$x = 18$$

$$\frac{d^2A}{dx^2} = -2\pi \text{ (Negative)}$$

Hence, the maximum value of A

$= 36\pi(18) - \pi(18)^2$

$= 648\pi - 324\pi$

$= 324\pi$ cm^2

21 $V = \pi r^2 h$

$$\pi r^2 h = 686\pi$$
$$r^2 h = 686$$
$$h = \frac{686}{r^2}$$

$$A = 2\pi r^2 + 2\pi rh$$
$$A = 2\pi r^2 + 2\pi r\left(\frac{686}{r^2}\right)$$
$$A = 2\pi r^2 + \frac{1\ 372\pi}{r} \text{ [Shown]}$$
$$\frac{dA}{dr} = 4\pi r - \frac{1\ 372\pi}{r^2}$$

When A has a stationary value,

$$\frac{dA}{dr} = 0$$
$$4\pi r - \frac{1\ 372\pi}{r^2} = 0$$
$$4r = \frac{1\ 372}{r^2}$$
$$r^3 = 343$$
$$r = 7$$

$$\frac{d^2A}{dr^2} = 4\pi + \frac{2\ 744}{r^3} \text{ (Positive)}$$

Hence, the minimum value of A

$= 2\pi(7)^2 + \frac{1\ 372\pi}{7}$

$= 294\pi$ cm^2

22 (a) Perimeter = 100

$$2x + 2y = 100$$
$$x + y = 50$$
$$x = 50 - y$$

A = Area of rectangle – Area of the two semicircles

$$A = xy - \pi r^2$$
$$A = y(50 - y) - \pi\left(\frac{y}{2}\right)^2$$
$$A = 50y - y^2 - \frac{\pi y^2}{4}$$
$$A = 50y - \frac{4y^2 + \pi y^2}{4}$$
$$A = 50y - \left(\frac{4+\pi}{4}\right)y^2 \text{ [Shown]}$$

(b) $\frac{dA}{dy} = 50 - 2y\left(\frac{4+\pi}{4}\right) = 50 - y\left(\frac{4+\pi}{2}\right)$

When A has a stationary value,

$$\frac{dA}{dy} = 0$$

$$50 - y\left(\frac{4+\pi}{2}\right) = 0$$

$$y\left(\frac{4+\pi}{2}\right) = 50$$

$$y = \frac{100}{4+\pi}$$

Width = $y = \frac{100}{4+\pi}$ cm

$$x = 50 - y$$

$$x = 50 - \left(\frac{100}{4+\pi}\right)$$

$$x = \frac{50(4+\pi) - 100}{4+\pi}$$

$$x = \frac{200 + 50\pi - 100}{4+\pi}$$

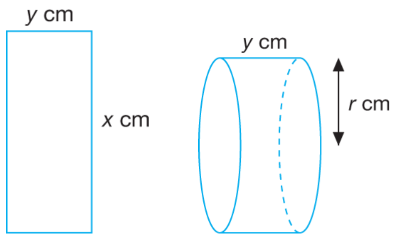
$$x = \frac{50\pi + 100}{4+\pi}$$

Length = $x = \frac{50\pi + 100}{4+\pi}$ cm

$\frac{d^2A}{dy^2} = -\left(\frac{4+\pi}{2}\right)$ [Negative]

Hence, the value of A is a maximum.

23



Perimeter of the rectangle = 50 cm

$$2x + 2y = 50$$

$$x + y = 25$$

$$y = 25 - x$$

Perimeter of the right part of the cylinder is equal to the length of the rectangle

$$2\pi r = x$$

$$r = \frac{x}{2\pi}$$

Volume of the cylinder

$$V = \pi r^2 h$$

$$V = \pi\left(\frac{x}{2\pi}\right)^2 y$$

$$V = \pi\left(\frac{x^2}{4\pi^2}\right)(25 - x)$$

$$V = \frac{1}{4\pi}\left(25x^2 - x^3\right)$$

$$\frac{dV}{dx} = \frac{1}{4\pi}\left(50x - 3x^2\right)$$

$$\frac{dV}{dx} = \frac{1}{4\pi}x(50 - 3x)$$

When V has a stationary value,

$$\frac{dV}{dx} = 0$$

$$\frac{1}{4\pi}x(50 - 3x) = 0$$

$$x = \frac{50}{3}$$

$$y = 25 - \frac{50}{3} = \frac{25}{3}$$

$$\frac{d^2V}{dx^2} = \frac{1}{4\pi}(50 - 6x)$$

When $x = \frac{50}{3}$,

$$\frac{d^2V}{dx^2} = \frac{1}{4\pi}\left[50 - 6\left(\frac{50}{3}\right)\right] = -\frac{25}{2\pi} \ (< 0)$$

Hence, the volume of the cylinder is a maximum.

Length = $16\frac{2}{3}$ cm

Width = $8\frac{1}{3}$ cm

24 V = Area of triangle $ABC \times CD$

$$V = \frac{1}{2}(x)(10) \times 5x$$

$$V = 5x(5x)$$

$$V = 25x^2$$

$$\frac{dV}{dx} = 50x$$

$$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$$

$$= 50x \times 0.02$$

$$= 50(5) \times 0.02$$

$= 5 \text{ cm}^3 \text{ s}^{-1}$

25 $A = 2\pi r^2 + 2\pi r(12)$

$A = 2\pi r^2 + 24\pi r$

$\frac{dA}{dr} = 4\pi r + 24\pi$

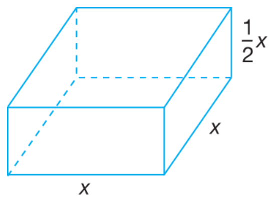
$\frac{dA}{dt} = \frac{dA}{dr} \times \frac{dr}{dt}$

$\frac{dA}{dt} = (4\pi r + 24\pi) \times 0.1$

$\frac{dA}{dt} = [4\pi(4) + 24\pi] \times 0.1$

$\frac{dA}{dt} = 4\pi \text{ cm}^2 \text{ s}^{-1}$

26 $V = x^2\left(\frac{1}{2}x\right)$



$V = \frac{1}{2}x^3$

$\frac{dV}{dx} = \frac{3}{2}x^2$

When $A = 2x^2 \times 4\left(\frac{1}{2}x\right)(x) = 4x^2$

When $A = 1\,600$,

$4x^2 = 1\,600$

$x^2 = 400$

$x = 20$

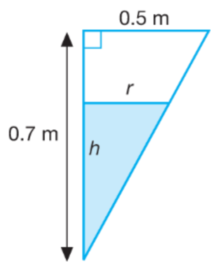
$\frac{dV}{dt} = \frac{dV}{dx} \times \frac{dx}{dt}$

$= \frac{3}{2}x^2 \times 0.02$

$= \frac{3}{2}(20)^2 \times 0.02$

$= 12 \text{ cm}^3 \text{ s}^{-1}$

27



Using the ratios of similar triangles,

$\frac{r}{0.5} = \frac{h}{0.7}$

$r = \frac{h}{0.7} \times 0.5$

$r = \frac{5}{7}h$

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi\left(\frac{5}{7}h\right)^2 h$

$V = \frac{1}{3}\pi\left(\frac{25}{49}h^2\right)h$

$V = \frac{25}{147}\pi h^3$

$\frac{dV}{dh} = \frac{25}{49}\pi h^2$

$\frac{dh}{dt} = \frac{dh}{dV} \times \frac{dV}{dt}$

$= \frac{49}{25\pi h^2} \times 0.1$

$= \frac{49}{25(3.142)(0.3)^2} \times 0.1$

$= 0.693 \text{ cm s}^{-1}$

28 (a) $m = 5x - 2$

$\frac{dm}{dx} = 5$

$\frac{dx}{dt} = \frac{dx}{dm} \times \frac{dm}{dt}$

$= \frac{1}{5} \times 2$

$= \frac{2}{5} \text{ unit s}^{-1}$

(b) $m = 5x - 2$ $\qquad y = -\frac{4}{m^2}$

$\frac{dm}{dx} = 5$ $\qquad \frac{dy}{dm} = \frac{8}{m^3}$

$x = \frac{m+2}{5}$

$\frac{dy}{dx} = \frac{\frac{dy}{dm}}{\frac{dx}{dm}} = \frac{\frac{8}{m^3}}{\frac{1}{5}} = \frac{40}{m^3} = \frac{40}{(5x-2)^3}$

(c) $\frac{\delta y}{\delta x} \approx \frac{dy}{dx}$

$\delta y \approx \frac{dy}{dx} \times \delta x$

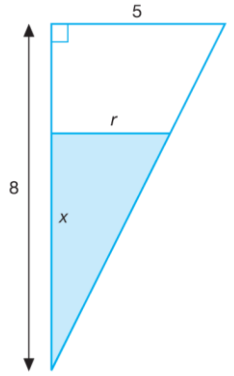
$\approx \frac{40}{(5x-2)^3} \times \left(\frac{4}{5} - 1\right)$

$\approx \frac{40}{(5\times1-2)^3} \times \left(-\frac{1}{5}\right)$

$\approx -\frac{40}{27} \times \frac{1}{5}$

$\approx -\frac{8}{27}$

29 (a)



Using the ratios of similar triangles,

$\frac{r}{5} = \frac{x}{8}$

$r = \frac{5x}{8}$

$V = \frac{1}{3}\pi r^2 h$

$V = \frac{1}{3}\pi \left(\frac{5x}{8}\right)^2 (x)$

$V = \frac{25}{192}\pi x^3$ [Shown]

(b) $\frac{\delta V}{\delta x} \approx \frac{dV}{dx}$

$\delta V \approx \frac{dV}{dx} \times \delta x$

$\approx \frac{25}{64}\pi x^2 \times (4.08-4)$

$\approx \frac{25}{64}\pi (4)^2 (0.08)$

$\approx 0.5\pi \text{ cm}^3$

30 ` $y = \frac{27}{x^4}$

$\frac{dy}{dx} = \frac{-108}{x^5}$

When $x = 3$

$\frac{dy}{dx} = \frac{-108}{3^5} = -\frac{4}{9}$

(a) $y_{new} = y_{originall} + \frac{dy}{dx} \times \delta x$

$\frac{27}{2.99^4} = \frac{27}{3^4} + \left(-\frac{4}{9}\right)(2.99-3)$

$= 0.3378$

(b) $\frac{27}{3.01^4} = \frac{27}{3^4} + \left(-\frac{4}{9}\right)(3.01-3)$

$= 0.3289$

SPM Spot

1 $y = \dfrac{p}{2 + qx}$

$y = p(2 + qx)^{-1}$

$\dfrac{dy}{dx} = -p(2 + qx)^{-2}(q)$

$= \dfrac{-pq}{(2 + qx)^2}$

At (1, 1), the gradient of the curve is $\dfrac{3}{5}$.

$\dfrac{-pq}{[2 + q(1)]^2} = \dfrac{3}{5}$

$-5pq = 3(4 + 4q + q^2)$ … (1)

The curve passes through the point (1, 1).

$1 = \dfrac{p}{2 + q(1)}$

$p = q + 2$ … (2)

Substitute (2) into (1) :

$-5q(2 + q) = 3(4 + 4q + q^2)$

$-10q - 5q^2 = 12 + 12q + 3q^2$

$8q^2 + 22q + 12 = 0$

$4q^2 + 11q + 6 = 0$

$(q + 2)(4q + 3) = 0$

$q = -2$ or $q = -\dfrac{3}{4}$

From (2) :

When $q = -2$,

$p = -2 + 2$

$p = 0$

When $q = -\dfrac{3}{4}$,

$p = -\dfrac{3}{4} + 2$

$p = 1\dfrac{1}{4}$

2 (a) Volume of the cone, V

$= \frac{1}{3}\pi r^2(12)$

$= 4\pi r^2$

$\frac{dV}{dr} = 8\pi r$

$\frac{\delta V}{\delta r} \approx \frac{dV}{dr}$

$\delta V \approx \frac{dV}{dr} \times \delta r$

$\approx 8\pi r \times (3.9 - 4)$

$\approx 8\pi(4)(-0.1)$

$\approx -3.2\pi \text{ cm}^3$

(b) Volume of the combined solid, V

$= 2x(2x)(x) + \pi(x)^2 y$

$= 4x^3 + \pi x^2 y$

Given that $V = 27$,

$4x^3 + \pi x^2 y = 27$

$\pi x^2 y = 27 - 4x^3$

$y = \frac{27 - 4x^3}{\pi x^2}$

$A = (2x)(2x) + 4(2x^2) + [(2x)(2x) - \cancel{\pi x^2}] + 2\pi x y + \cancel{\pi x^2}$

$A = 16x^2 + 2\pi x\left(\frac{27 - 4x^3}{\pi x^2}\right)$

$A = 16x^2 + \frac{2}{x}(27 - 4x^3)$

$A = 16x^2 + \frac{54}{x} - 8x^2$

$A = 8x^2 + \frac{54}{x}$ [Shown]

$\frac{dA}{dx} = 16x - \frac{54}{x^2}$

When A has a stationary value,

$\frac{dA}{dx} = 0$

$16x - \frac{54}{x^2} = 0$

$\frac{54}{x^2} = 16x$

$x^3 = \frac{27}{8}$

$x = \frac{3}{2}$

$\frac{d^2A}{dx^2} = 16 + \frac{108}{x^3}$ (Positive)

Hence, the value of A is a minimum.

3 (a) (i) Let the area of *PQRST* = *L* cm^2

$$L = x(26-2x) + \frac{1}{2}(2x)(x)$$
$$L = 26x - 2x^2 + x^2$$
$$L = 26x - x^2$$

(ii) $\frac{dL}{dx} = 26 - 2x$

When *L* has a stationary value,

$$\frac{dL}{dx} = 0$$
$$26 - 2x = 0$$
$$2x = 26$$
$$x = 13$$

$\frac{d^2L}{dx^2} = -2$ (Negative)

Hence, the maximum area of *PQRST*

$= 26(13) - (13)^2$

$= 169$ cm^2

(b) $\frac{dL}{dx} = 26 - 2x$

$$\frac{dx}{dL} = \frac{1}{26-2x}$$

When $x = 5$,

$$\frac{dx}{dA} = \frac{1}{26-2(5)}$$

$$\frac{dx}{dt} = \frac{dx}{dL} \times \frac{dL}{dt}$$
$$= \frac{1}{26-2(5)} \times 36$$

$= 2.25$ cm s^{-1}

Hence, the rate of change of *QS* is 2.25 cm s^{-1}.

(c) $\frac{\delta L}{\delta x} \approx \frac{dL}{dx}$

$$\delta L \approx \frac{dL}{dx} \times \delta x$$
$$\approx (26-2x) \times (1.98-2)$$
$$\approx [26-2(2)] \times (1.98-2)$$
$$\approx -0.44 \text{ cm}^2$$

Hence, the approximate change in the area, of *PQRST* is –0.44 cm^2.

