Form 4 Chapter 1
Functions
Fully-Worked Solutions

UPSKILL 1.1a

1 (a) Each image is obtained by changing the sign of each object. Hence, $f(x) = -x$.

(b) (i) Domain = {6, 7, 8}

(ii) Codomain = {–6, –7, –8, –9}

(iii) Range = {–6, –7, –8}

(iv) The object of –6 is 6.

(v) The image of 7 is –7.

2 (a) Since the vertical line intersects the graph only once, then it is a function.

(b) Since the vertical line intersects the graph twice, then it is not a function.

3 $2x - 1 \neq 0$

$x \neq \dfrac{1}{2}$

But it is given that $x \neq h$.

Hence by comparison, $h = \dfrac{1}{2}$.

UPSKILL 1.1b

1 (a) Domain = $-4 \leq x \leq 4$

Range = $1 \leq f(x) \leq 5$

(b) Domain = $-1 \leq x \leq 2$

Range = $0 \leq f(x) \leq 9$

UPSKILL 1.1c

1 (a) $f(x) = \dfrac{18}{2x - 9}$

(i) $f(0) = \dfrac{18}{2(0) - 9} = -2$

(ii) $f(3) = \dfrac{18}{2(3) - 9} = \dfrac{18}{-3} = -6$

(b) (i) $f(x) = 2$

$$\frac{18}{2x - 9} = 2$$
$$18 = 4x - 18$$
$$4x = 36$$
$$x = 9$$

(ii) $f(x) = 6$

$$\frac{18}{2x - 9} = 6$$
$$18 = 12x - 54$$
$$12x = 72$$
$$x = 6$$

2 (a) $f(3) = -5$

$$\frac{a}{3 - b} = -5$$
$$a = -15 + 5b \quad \ldots \text{①}$$

$f(-5) = -1$

$$\frac{a}{-5 - b} = -1$$
$$a = 5 + b \quad \ldots \text{②}$$

Substitute ② into ①:

$$5 + b = -15 + 5b$$
$$4b = 20$$
$$b = 5$$

From ②: $a = 5 + 5 = 10$

(b) $f(x) = \dfrac{10}{x - 5}$

$$x - 5 \neq 0$$
$$x \neq 5$$

Hence, the value of x such that f is undefined is 5.

3 (a) $f(x) = \dfrac{px + q}{x - 2}$

$f(3) = 4$

$$\frac{3p + q}{3 - 2} = 4$$
$$3p + q = 4$$
$$q = 4 - 3p \quad \ldots \text{①}$$

$f(1) = 2$

$$\frac{p + q}{1 - 2} = 2$$
$$p + q = -2 \quad \ldots \text{②}$$

Substitute ① into ②:

$$p + 4 - 3p = -2$$
$$-2p = -6$$
$$p = 3$$

From ①:

$q = 4 - 3(3) = -5$

(b) $f(x)=\frac{3x-5}{x-2}$

The value of x such that f is undefined is 2.

(c) $f(x)=\frac{4}{3}x$

$$\frac{3x-5}{x-2}=\frac{4x}{3}$$
$$9x-15=4x^2-8x$$
$$4x^2-17x+15=0$$
$$(x-3)(4x-5)=0$$
$$x=3 \text{ or } x=\frac{5}{4}$$

4 (a) $g(x)=ax+\frac{b}{x}$

$$g(2)=7$$
$$2a+\frac{b}{2}=7$$
$$4a+b=14$$
$$b=14-4a \;\ldots \text{①}$$

$$g(-1)=-5$$
$$-a+\frac{b}{-1}=-5$$
$$-a-b=-5 \;\ldots \text{②}$$

Substitute ① into ②:

$$-a-(14-4a)=-5$$
$$-a-14+4a=-5$$
$$3a=9$$
$$a=3$$

From ①: $b=14-4a$

$b=14-4(3)=2$

(b) $g(x)=3x+\frac{2}{x}$

g is undefined when $x = 0$.

(c) $g(x)=7$

$$3x+\frac{2}{x}=7$$
$$3x^2+2=7x$$
$$3x^2-7x+2=0$$
$$(3x-1)(x-2)=0$$
$$x=\frac{1}{3} \text{ or } x=2$$

$x=2$ is not accepted.

$\therefore x=\frac{1}{3}$

5 (a) $g(x)=a+bx$

$$g(1)=-3$$
$$a+b=-3$$
$$a=-3-b \;\ldots \text{①}$$

$$g(-2)=3$$
$$a-2b=3 \;\ldots \text{②}$$

Substitute ① into ②:

$$-3-b-2b=3$$
$$-3-3b=3$$
$$-3b=6$$
$$b=-2$$

From ①: $a=-3-(-2)=-1$

(b) $g(x)=-1-2x$

$$g(n^2+1)=5n-6$$
$$-1-2(n^2+1)=5n-6$$
$$-2n^2-2-1-5n+6=0$$
$$-2n^2-5n+3=0$$
$$2n^2+5n-3=0$$
$$(2n-1)(n+3)=0$$
$$n=\frac{1}{2} \text{ or } n=-3$$

6 $f(x)=x$

$$\frac{5x-4}{x+1}=x$$
$$5x-4=x^2+x$$
$$x^2-4x+4=0$$
$$(x-2)^2=0$$
$$x=2$$

7 $f(x)=x$

$$\frac{12}{x-4}=x$$
$$12=x^2-4x$$
$$x^2-4x-12=0$$
$$(x-6)(x+2)=0$$
$$x=6 \text{ or } -2$$

8 (a) $f(x)=px+qx^2$

$$f(-1)=-5$$
$$-p+q=-5$$
$$q=p-5 \;\ldots \text{①}$$

$f(-2) = -16$
$-2p + 4q = -16$
$p - 2q = 8 \quad \ldots \text{②}$

Substitute ① into ②:
$p - 2(p - 5) = 8$
$p - 2p + 10 = 8$
$-p = -2$
$p = 2$

From ①: $q = 2 - 5 = -3$

(b) $f(x) = 2x - 3x^2$
$f(x) = x$
$2x - 3x^2 = x$
$3x^2 - x = 0$
$x(3x - 1) = 0$
$x = 0$ or $\frac{1}{3}$

UPSKILL 1.1d

1 $h(x) = |x^2 - 4x - 3|$

(a) $h(-3) = |(-3)^2 - 4(-3) - 3|$
$= |18|$
$= 18$

(b) $h(0) = |0^2 - 4(0) - 3| = |-3| = 3$

(c) $h(2) = |2^2 - 4(2) - 3| = |-7| = 7$

2 (a) (i) $f(2)$
$= |2 - 5(2)|$
$= |-8|$
$= 8$

(ii) $f(-2)$
$= |2 - 5x|$
$= |2 - 5(-2)|$
$= 12$

(b) $f(x) = 7$
$|2 - 5x| = 7$
$2 - 5x = \pm 7$

$2 - 5x = 7$
$5x = -5$
$x = -1$

$2 - 5x = -7$
$-5x = -9$
$x = \frac{9}{5}$

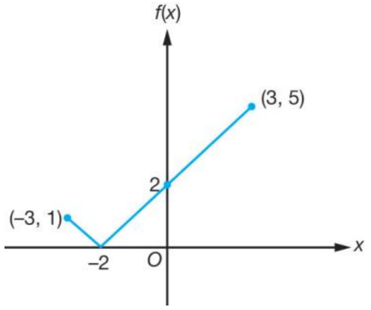
3 $g(x) = 7$
$|2x + 1| = 7$
$2x + 1 = \pm 7$

$2x + 1 = 7$
$2x = 6$
$x = 3$

$2x + 1 = -7$
$2x = -8$
$x = -4$

4 (a) $f(x) = |x + 2|$

x	–3	–2	–1	0	1	2	3
$f(x)$	1	0	1	2	3	4	5



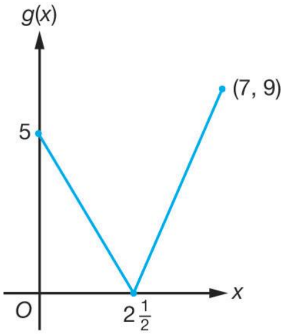
The range of $f(x)$ is $0 \leq f(x) \leq 5$.

(b) $g(x) = |2x - 5|$

x	0	1	2	3	4	5	6	7
$g(x)$	5	3	1	1	3	5	7	9

When $|2x - 5| = 0$
$2x - 5 = 0$
$x = \frac{5}{2} = 2\frac{1}{2}$

The graph touches the x-axis at $\left(2\frac{1}{2}, 0\right)$.



The range of $g(x)$ is $0 \le g(x) \le 9$.

(c) $h(x) = |3 - 2x|$

x	–3	–2	–1	0	1	2	3	4
$h(x)$	9	7	5	3	1	1	3	5

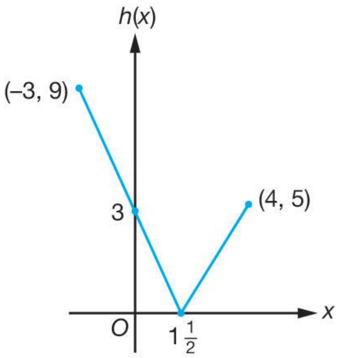
On the x-axis, $y = 0$,

$|3 - 2x| = 0$

$3 - 2x = 0$

$x = 1\frac{1}{2}$

The graph touches the x-axis at $\left(1\frac{1}{2}, 0\right)$.



The range of $h(x)$ is $0 \le h(x) \le 9$.

5 (a) $|h(x)| < 3$

$|4x - 3| < 3$

$-3 < 4x - 3 < 3$

The first inequality is

$-3 < 4x - 3$

$4x > 0$

$x > 0 \ldots ①$

The second inequality is

$4x - 3 < 3$

$4x < 6$

$x < \frac{3}{2} \ldots ②$

Combining ① and ②:

$0 < x < \frac{3}{2}$

(b) $|h(x)| > 1$

$|4x - 3| > 1$

The first inequality is

$4x - 3 < -1$

$4x < 2$

$x < \frac{1}{2} \ldots ①$

The second inequality is

$4x - 3 > 1$

$4x > 4$

$x > 1 \ldots ②$

Combining ① and ②:

$x < \frac{1}{2}$ or $x > 1$.

UPSKILL 1.2a

1 $f(x) = |4 - 5x|$

$g(x) = \sqrt{x - 2}$

(a) $fg(6) = f\left(\sqrt{6 - 2}\right)$

$= f(2)$

$= |4 - 5(2)|$

$= |-6|$

$= 6$

(b) $gf(2) = g\left(|4 - 5(2)|\right)$

$= g\left(|-6|\right)$

$= g(6)$

$= \sqrt{6 - 2}$

$= \sqrt{4}$

$= 2$

(c) $f^2(0) = ff(0)$
$= f(|4-5(0)|)$
$= f(4)$
$= |4-5(4)|$
$= |-16|$
$= 16$

(d) $g^2(51) = gg(51)$
$= g(\sqrt{51-2})$
$= g(\sqrt{49})$
$= g(7)$
$= \sqrt{7-2}$
$= \sqrt{5}$
$= 2.236$

2 (a) $fg(x) = f[g(x)]$
$= f(3x+1)$
$= (3x+1)^2 - 1$
$= 9x^2 + 6x + 1 - 1$
$= 9x^2 + 6x$

$gf(x) = g(x^2-1)$
$= 3(x^2-1)+1$
$= 3x^2 - 3 + 1$
$= 3x^2 - 2$

(b) $fg(x) = f(1-3x)$
$= (1-3x+1)^2$
$= (2-3x)^2$
$= 4 - 12x + 9x^2$

$gf(x) = g[(x+1)^2]$
$= 1 - 3(x+1)^2$
$= 1 - 3(x^2+2x+1)$
$= -2 - 3x^2 - 6x$
$= -3x^2 - 6x - 2$

(c) $fg(x) = f\left(\frac{1}{x^2+2}\right)$
$= 2 - \left(\frac{1}{x^2+2}\right)$
$= \frac{2(x^2+2)-1}{x^2+2}$
$= \frac{2x^2+3}{x^2+2}$

$gf(x) = g(2-x)$
$= \frac{1}{(2-x)^2+2}$
$= \frac{1}{4-4x+x^2+2}$
$= \frac{1}{x^2-4x+6}$

3 (a) $f^2(x) = ff(x)$
$= f(4x-3)$
$= 4(4x-3)-3$
$= 16x - 12 - 3$
$= 16x - 15$

(b) $g^2(x) = gg(x)$
$= g(x+1)$
$= (x+1)+1$
$= x+2$

$f^2(x) = g^2(x)$
$16x - 15 = x + 2$
$15x = 17$
$x = \frac{17}{15}$

4 (a) $f^2(x) = ff(x)$
$= f\left(\frac{x-1}{x+1}\right)$
$= \frac{\left(\frac{x-1}{x+1}\right)-1}{\left(\frac{x-1}{x+1}\right)+1}$
$= \frac{\frac{x-1-(x+1)}{x+1}}{\frac{x-1+(x+1)}{x+1}}$
$= \frac{-2}{2x}$
$= -\frac{1}{x}, \ x \neq 0$

(b) $f^4(x) = f^2 f^2(x)$

$= f^2\left(-\frac{1}{x}\right)$

$= -\frac{1}{\left(-\frac{1}{x}\right)}$

$= x$

(c) $f^{12}(x) = f^4 f^4 f^4(x)$

$= f^4 f^4(x)$

$= f^4(x)$

$= x$

(d) $f^{13}(x) = f\, f^{12}(x)$

$= f(x)$

$= \frac{x-1}{x+1}, \quad x \neq -1$

5 (a) $f(8) = 4$

(b) $g(4) = 16$

(c) $gf(8) = g(4) = 16$

6 The function that maps x straight away to z is nm.

$nm(x) = n(3x+2)$

$= (3x+2)^2 - 10$

$= 9x^2 + 12x + 4 - 10$

$= 9x^2 + 12x - 6$

7 (a) $f(x) = 5x + 6,\ g(x) = 2x - 1$

$gf(x) = g[f(x)]$

$= g(5x+6)$

$= 2(5x+6) - 1$

$= 10x + 11$

When $gf(x) = 9$,

$10x + 11 = 9$

$10x = -2$

$x = -\frac{1}{5}$

(b) $g(x) = \frac{6}{x-2}$

$g^2(x) = gg(x)$

$= g\left(\frac{6}{x-2}\right)$

$= \frac{6}{\left(\frac{6}{x-2}\right) - 2}$

$= \frac{6}{\frac{6-2(x-2)}{x-2}}$

$= \frac{6(x-2)}{10-2x}$

$= \frac{3(x-2)}{5-x}$

When

$g^2(x) = -1$,

$\frac{3(x-2)}{5-x} = -1$

$3x - 6 = x - 5$

$2x = 1$

$x = \frac{1}{2}$

8 (a) $fg(x) = f(x-2)$

$= [(x-2)+1]^2$

$= (x-1)^2$

$= x^2 - 2x + 1$

$gf(x) = g[(x+1)^2]$

$= (x+1)^2 - 2$

$= x^2 + 2x - 1$

(b) (i) $fg(x) = 4$

$x^2 - 2x + 1 = 4$

$x^2 - 2x - 3 = 0$

$(x+1)(x-3) = 0$

$x = -1$ or 3

(ii) $gf(x) = 7$

$x^2 + 2x - 1 = 7$

$x^2 + 2x - 1 - 7 = 0$

$x^2 + 2x - 8 = 0$

$(x-2)(x+4) = 0$

$x = 2$ or -4

(c) $fg(x) = gf(x)$

$x^2 - 2x + 1 = x^2 + 2x - 1$

$-4x = -2$

$x = \frac{1}{2}$

UPSKILL 1.2b

1 (a)
$$fg(x) = 3x - 2$$
$$f[g(x)] = 3x - 2$$
$$g(x) + 2 = 3x - 2$$
$$g(x) = 3x - 4$$

(b)
$$fg(x) = \frac{2x+5}{x-2}$$
$$f[g(x)] = \frac{2x+5}{x-2}$$
$$3[g(x)] + 2 = \frac{2x+5}{x-2}$$
$$3g(x) = \frac{2x+5}{x-2} - 2$$
$$3g(x) = \frac{2x+5-2(x-2)}{x-2}$$
$$3g(x) = \frac{9}{x-2}$$
$$g(x) = \frac{3}{x-2}, \quad x \neq 2$$

(c)
$$fg(x) = x^2 + 4x + 3$$
$$f[g(x)] = x^2 + 4x + 3$$
$$[g(x)]^2 - 1 = x^2 + 4x + 3$$
$$[g(x)]^2 = x^2 + 4x + 3 + 1$$
$$[g(x)]^2 = x^2 + 4x + 4$$
$$[g(x)]^2 = (x+2)^2$$
$$g(x) = x + 2$$

2 (a)
$$gf(x) = \frac{3}{x-2}$$
$$g[f(x)] = \frac{3}{x-2}$$
$$g(x+1) = \frac{3}{x-2}$$
Let $x + 1 = u$
$$x = u - 1$$
$$g(u) = \frac{3}{(u-1)-2}$$
$$g(u) = \frac{3}{u-3}$$
$$g(x) = \frac{3}{x-3}, \quad x \neq 3$$

(b)
$$gf(x) = \frac{5}{10x-1}$$
$$g[f(x)] = \frac{5}{10x-1}$$
$$g\left(\frac{1}{x}\right) = \frac{5}{10x-1}$$
Let $\frac{1}{x} = u$
$$x = \frac{1}{u}$$
$$g(u) = \frac{5}{10\left(\frac{1}{u}\right) - 1}$$
$$g(u) = \frac{5u}{10-u}$$
$$g(x) = \frac{5x}{10-x}, \quad x \neq 10$$

(c)
$$gf(x) = 9x^2 + 9x + 2$$
$$g[f(x)] = 9x^2 + 9x + 2$$
$$g[3x+2] = 9x^2 + 9x + 2$$
Let $3x + 2 = u$
$$x = \frac{u-2}{3}$$
$$g(u) = 9\left(\frac{u-2}{3}\right)^2 + 9\left(\frac{u-2}{3}\right) + 2$$
$$g(u) = 9\left(\frac{u^2-4u+4}{9}\right) + 3(u-2) + 2$$
$$g(u) = u^2 - 4u + 4 + 3u - 6 + 2$$
$$g(u) = u^2 - u$$
$$g(x) = x^2 - x$$

3 (a)
$$fg(x) = 4x - 12$$
$$f[g(x)] = 4x - 12$$
$$2g(x) = 4x - 12$$
$$g(x) = 2x - 6$$

$$hf(x) = \frac{2x+1}{2}$$
$$h[f(x)] = \frac{2x+1}{2}$$
$$h(2x) = \frac{2x+1}{2}$$
Let $2x = u$
$$x = \frac{u}{2}$$

$$h(u)=\frac{2\left(\frac{u}{2}\right)+1}{2}$$
$$h(u)=\frac{u+1}{2}$$
$$h(x)=\frac{x+1}{2}$$

(b) $$gf(x)=\frac{2x-1}{3}$$
$$g[f(x)]=\frac{2x-1}{3}$$
$$g(2x-2)=\frac{2x-1}{3}$$
Let $2x-2=u$
$$x=\frac{u+2}{2}$$
$$g(u)=\frac{2\left(\frac{u+2}{2}\right)-1}{3}$$
$$g(u)=\frac{u+2-1}{3}$$
$$g(u)=\frac{u+1}{3}$$
$$g(x)=\frac{x+1}{3}$$

$$fh(x)=2x^2$$
$$f[h(x)]=2x^2$$
$$2h(x)-2=2x^2$$
$$h(x)-1=x^2$$
$$h(x)=x^2+1$$

(c) $$fg(x)=x^2+6x+7$$
$$f[g(x)]=x^2+6x+7$$
$$[g(x)]^2-2=x^2+6x+7$$
$$[g(x)]^2=x^2+6x+9$$
$$[g(x)]^2=(x+3)^2$$
$$g(x)=x+3$$

$$hf(x)=2x^2-7$$
$$h[f(x)]=2x^2-7$$
$$h(x^2-2)=2x^2-7$$
Let $x^2-2=u$
$$x^2=u+2$$
$$h(u)=2(u+2)-7$$
$$h(u)=2u-3$$
$$h(x)=2x-3$$

UPSKILL 1.2c

1 $gf(x)=g[f(x)]$
$=g(1-x)$
$=p(1-x)^2+h$
$=p(1-2x+x^2)+h$
$=p-2px+px^2+h$
$=px^2-2px+p+h$

But it is given that $gf(x)=3x^2-6x+5$.
By comparison,
$p=3$ and $p+h=5$
$3+h=5$
$h=2$

2 (a) $f(x)=hx+k$
$f^2(x)=ff(x)$
$=f(hx+k)$
$=h(hx+k)+k$
$=h^2x+hk+k$

But it is given that $f^2(x)=81x-16$.

By comparison,
$h^2=81$
$h=\pm 9$

$hk+k=-16$
When $h=9$,
$9k+k=-16$
$10k=-16$
$$k=-\frac{8}{5}$$

When $h=-9$,
$-9k+k=-16$
$-8k=-16$
$k=2$

(b) When $h=-9$ and $k=2$,
$f(x)=-9x+2$

$f(x^2)=3x$
$-9x^2+2=3x$
$9x^2+3x-2=0$
$(3x-1)(3x+2)=0$

$$x = \frac{1}{3} \text{ or } x = -\frac{2}{3}$$

3 The composite function required is $cf(t)$.

$$cf(t) = c[f(t)] = \frac{3}{4}[2t] = \frac{3}{2}t$$

UPSKILL 1.3

1 Let $f^{-1}(4) = y$

$f(y) = 4$

(a) $3 - 2y = 4$

$$y = \frac{1}{-2}$$

$$f^{-1}(4) = -\frac{1}{2}$$

(b) $6 - \frac{5}{y} = 4$

$$\frac{5}{y} = 2$$

$$y = \frac{5}{2}$$

$$f^{-1}(4) = \frac{5}{2}$$

(c) $\frac{3y+2}{2y+3} = 4$

$3y + 2 = 8y + 12$

$5y = -10$

$y = -2$

$f^{-1}(4) = -2$

2 (a) The horizontal line intersects the curve at only one point. Hence, $f(x) = \frac{2x-1}{x+2}$, $x \neq -2$ has inverse function.

(b) The horizontal line intersects the curve at more than one point. Hence, $f(x) = x^2 - 5x + 6$, $x \neq -2$ does not have inverse function.

3 (a) Let $f^{-1}(x) = y$

$f(y) = x$

$5 - 4y = x$

$4y = 5 - x$

$$y = \frac{5-x}{4}$$

$$f^{-1}(x) = \frac{5-x}{4}$$

(b) Let $g^{-1}(x) = y$

$g(y) = x$

$$\frac{3y-4}{2} = x$$

$3y - 4 = 2x$

$$y = \frac{2x+4}{3}$$

$$g^{-1}(x) = \frac{2x+4}{3}$$

(c) Let $h^{-1}(x) = y$

$h(y) = x$

$$9 - \frac{3}{y} = x$$

$$\frac{3}{y} = 9 - x$$

$$y = \frac{3}{9-x}$$

$$h^{-1}(x) = \frac{3}{9-x}, \ x \neq 9$$

(d) Let $m^{-1}(x) = y$

$m(y) = x$

$$\frac{2y+2}{5y-3} = x$$

$2y + 2 = 5xy - 3x$

$2y - 5xy = -3x - 2$

$y(2 - 5x) = -3x - 2$

$$y = \frac{-3x-2}{2-5x}$$

$$m^{-1}(x) = \frac{3x+2}{5x-2}, \ x \neq \frac{2}{5}$$

(e) Let $n^{-1}(x) = y$

$n(y) = x$

$\sqrt{2-y} = x$

$2 - y = x^2$

$y = 2 - x^2$

$n^{-1}(x) = 2 - x^2$

$f(x) = 5 - 4x$

$$f^{-1}(x) = \frac{5-x}{4}$$

$$ff^{-1}(x) = f\left(\frac{5-x}{4}\right)$$
$$= 5 - 4\left(\frac{5-x}{4}\right)$$
$$= 5 - (5 - x)$$
$$= x \text{ [Shown]}$$

$$f^{-1}f(x) = f^{-1}(5-4x)$$
$$= \frac{5-(5-4x)}{4}$$
$$= \frac{4x}{4}$$
$$= x \text{ [Shown]}$$

4 (a) Let $g^{-1}(x) = y$

$$g(y) = x$$
$$\frac{y-1}{y-2} = x$$
$$y - 1 = xy - 2x$$
$$y - xy = 1 - 2x$$
$$y(1-x) = 1 - 2x$$
$$y = \frac{1-2x}{1-x}$$
$$g^{-1}(x) = \frac{1-2x}{1-x}, \ x \neq 1$$

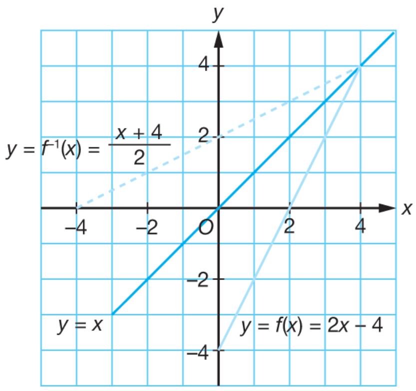
(b) $$gg^{-1}(x) = g\left(\frac{1-2x}{1-x}\right)$$
$$= \frac{\left(\frac{1-2x}{1-x}\right) - 1}{\left(\frac{1-2x}{1-x}\right) - 2}$$
$$= \frac{\frac{1-2x-(1-x)}{1-x}}{\frac{1-2x-2(1-x)}{1-x}}$$
$$= \frac{1-2x-1+x}{1-2x-2+2x}$$
$$= \frac{-x}{-1}$$
$$= x \text{ [Shown]}$$

$$g^{-1}g(x) = g^{-1}\left(\frac{x-1}{x-2}\right)$$
$$= \frac{1-2\left(\frac{x-1}{x-2}\right)}{1-\left(\frac{x-1}{x-2}\right)}$$
$$= \frac{\frac{x-2-2x+2}{x-2}}{\frac{x-2-x+1}{x-2}}$$
$$= \frac{-x}{-1}$$
$$= x \text{ [Shown]}$$

5 (a) Let $f^{-1}(x) = y$

$$f(y) = x$$
$$2y - 4 = x$$
$$y = \frac{x+4}{2}$$
$$f^{-1}(x) = \frac{x+4}{2}$$

(b)



The graph of f^{-1} is the reflection of the graph of f in the straight line $y = x$.

(c) (i) The domain of $f(x)$ is $0 \le x \le 4$.
The range of $f(x)$ is $-4 \le f(x) \le 4$.

(ii) The domain of $f^{-1}(x)$ is $-4 \le x \le 4$.
The range of $f^{-1}(x)$ is
$0 \le f^{-1}(x) \le 4$.

Conclusion
- The domain of $f^{-1}(x)$ is the range of $f(x)$.
- The range of $f^{-1}(x)$ is the domain of $f(x)$.

6 (a) $f(x) = \frac{3x-1}{x-2}, \ x \neq 2$

It is given that $x \neq h$.
By comparison, $h = 2$.

(b) $f^2(x) = ff(x) \quad f(x) = \frac{3x-1}{x-2}$

$$= f\left(\frac{3x-1}{x-2}\right)$$

$$=\frac{3\left(\frac{3x-1}{x-2}\right)-1}{\left(\frac{3x-1}{x-2}\right)-2}$$

$$=\frac{\frac{3(3x-1)-(x-2)}{x-2}}{\frac{3x-1-2(x-2)}{x-2}}$$

$$=\frac{9x-3-x+2}{3x-1-2x+4}$$

$$=\frac{8x-1}{x+3},\ x\neq -3$$

(c) Let $f^{-1}(x)=y$

$$f(y)=x$$

$$\frac{3y-1}{y-2}=x$$

$$3y-1=xy-2x$$

$$3y-xy=1-2x$$

$$y(3-x)=1-2x$$

$$y=\frac{1-2x}{3-x}$$

$$f^{-1}(x)=\frac{2x-1}{x-3},\ x\neq 3$$

7 $f(x)=\frac{4}{x}$

$g(x)=2x+3$

(a) Let $f^{-1}(x)=y$

$$f(y)=x$$

$$\frac{4}{y}=x$$

$$y=\frac{4}{x}$$

$$f^{-1}(x)=\frac{4}{x},\ x\neq 0$$

(b) Let $g^{-1}(x)=y$

$$g(y)=x$$

$$2y+3=x$$

$$y=\frac{x-3}{2}$$

$$g^{-1}(x)=\frac{x-3}{2}$$

(c) $f^{-1}g^{-1}=f^{-1}\left(\frac{x-3}{2}\right)$

$$=\frac{4}{\frac{x-3}{2}}$$

$$=\frac{8}{x-3},\ x\neq 3$$

(d) $g^{-1}f^{-1}(x)=g^{-1}\left(\frac{4}{x}\right)$

$$=\frac{\frac{4}{x}-3}{2}$$

$$=\frac{4-3x}{2x},\ x\neq 0$$

8 $f(x)=1-2x$

$g(x)=\frac{x+2}{x-2}$

(a) Let $f^{-1}(x)=y$

$$f(y)=x$$

$$1-2y=x$$

$$y=\frac{1-x}{2}$$

$$f^{-1}(x)=\frac{1-x}{2}$$

(b) Let $g^{-1}(x)=y$

$$g(y)=x$$

$$\frac{y+2}{y-2}=x$$

$$y+2=xy-2x$$

$$xy-y=2x+2$$

$$y(x-1)=2x+2$$

$$y=\frac{2x+2}{x-1}$$

$$g^{-1}(x)=\frac{2x+2}{x-1},\ x\neq 1$$

(c) $g^{-1}f^{-1}=g^{-1}\left(\frac{1-x}{2}\right)$

$$=\frac{2\left(\frac{1-x}{2}\right)+2}{\left(\frac{1-x}{2}\right)-1}$$

$$=\frac{3-x}{\frac{1-x-2}{2}}$$

$$= \frac{6-2x}{-x-1}$$
$$= \frac{2x-6}{x+1},\ x \neq -1$$

(d) $fg(x) = f\left(\frac{x+1}{x-2}\right)$
$$= 1-2\left(\frac{x+1}{x-2}\right)$$
$$= \frac{x-2-2(x+1)}{x-2}$$
$$= \frac{x-2-2x-4}{x-2}$$
$$= \frac{-x-6}{x-2}$$
$$fg(x) = \frac{x+6}{2-x},\ x \neq 2$$

(e) $y = \frac{x+6}{2-x}$
$$y(2-x) = x+6$$
$$2y - xy = x+6$$
$$x + xy = -2y-6$$
$$x(1+y) = 2y-6$$
$$x = \frac{2y-6}{y+1}$$
$$(fg)^{-1}(x) = \frac{2x-6}{x+1},\ x \neq -1$$
Yes, $(fg)^{-1}(x) = g^{-1}f^{-1}(x)$

9 (a) $f^2(x) = ff(x)$
$$= f(2x-1)$$
$$= 2(2x-1)-1$$
$$= 4x-3$$

Let $f^{-1}(x) = y$
$$f(y) = x$$
$$2y-1 = x$$
$$y = \frac{x+1}{2}$$
$$f^{-1}(x) = \frac{x+1}{2}$$

(b) $\left(f^{-1}\right)^2(x) = f^{-1}f^{-1}(x)$
$$= f^{-1}\left(\frac{x+1}{2}\right)$$
$$= \frac{\left(\frac{x+1}{2}\right)+1}{2}$$
$$= \frac{x+1+2}{4}$$
$$= \frac{x+3}{4}$$

$f^2(x) = ff(x)$
$$= f(2x-1)$$
$$= 2(2x-1)-1$$
$$= 4x-3$$

Let $y = 4x-3$
$$x = \frac{y+3}{4}$$
$$\left(f^2\right)^{-1}(x) = \frac{x+3}{4}$$

Hence, $\left(f^{-1}\right)^2(x) = \left(f^2\right)^{-1}(x)$

[Shown]

10 Let $f^{-1}(x) = y$
$$f(y) = x$$
$$\frac{y+p}{y-5} = x$$
$$y + p = xy - 5x$$
$$xy - y = p + 5x$$
$$y(x-1) = p+5x$$
$$y = \frac{5x+p}{x-1}$$
$$f^{-1}(x) = \frac{5x+p}{x-1},\ x \neq 1$$

But it is given that $f^{-1}(x) = \frac{qx+6}{x-1}$, $x \neq 1$.
Hence, by comparison, $q = 5$ and $p = 6$.

11 (a) Let $f^{-1}(x) = y$
$$f(y) = x$$
$$4y + h = x$$
$$y = \frac{x-h}{4}$$
$$f^{-1}(x) = \frac{x-h}{4}$$

But it is given that $f^{-1}(x) = \frac{x+5}{k}$.
Hence, by comparison,
$h = -5$ and $k = 4$.

(b) $f^{-1}f(b) = b^2 - 2$

$$b = b^2 - 2$$
$$b^2 - b - 2 = 0$$
$$(b-2)(b+1) = 0$$
$$b = 2 \text{ or } -1$$

12 (a) Let $f^{-1}(x) = y$

$$f(y) = x$$
$$\frac{3y-1}{y} = x$$
$$3y - 1 = xy$$
$$3y - xy = 1$$
$$y(3-x) = 1$$
$$y = \frac{1}{3-x}$$
$$f^{-1}(x) = \frac{1}{3-x}, \quad x \neq 3$$

But it is given that $f^{-1}(x) = \frac{1}{m-x}$.

Hence, by comparison, $m = 3$.

(b) $f^{-1}f(k^2+2) = (k+2)^2 + 2$

$$k^2 + 2 = k^2 + 4k + 4 + 2$$
$$4k + 4 = 0$$
$$k = -1$$

13 (a) Let $y = \frac{3x+8}{4}$

$$4y = 3x + 8$$
$$x = \frac{4y-8}{3}$$
$$f(x) = \frac{4x-8}{3}$$
$$f : x \rightarrow \frac{4x-8}{3}$$

(b) Let $y = \frac{2x+1}{x+4}$

$$y(x+4) = 2x + 1$$
$$xy + 4y = 2x + 1$$
$$xy - 2x = 1 - 4y$$
$$x(y-2) = 1 - 4y$$
$$x = \frac{1-4y}{y-2}$$
$$g(x) = \frac{1-4x}{x-2}, \quad x \neq 2$$
$$g : x \rightarrow \frac{1-4x}{x-2}, \quad x \neq 2$$

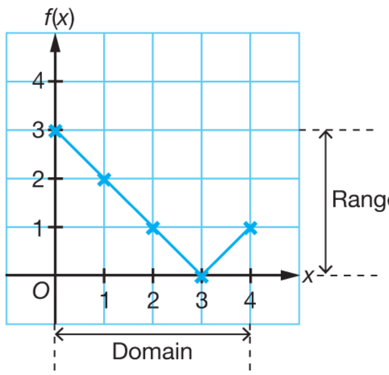
14 Let $V^{-1}(h) = y$

$$V(y) = h$$
$$900\pi y = h$$
$$y = \frac{h}{900\pi}$$
$$\therefore \quad V^{-1}(h) = \frac{h}{900\pi}$$

Summative Practice 1

1 (a)

x	0	1	2	3	4
$f(x)$	3	2	1	0	1



(b) The corresponding range of $f(x)$ is $0 \leq f(x) \leq 3$.

2

$$fg : x \rightarrow x^2 + 1$$
$$f[g(x)] = x^2 + 1$$
$$3g(x) - 7 = x^2 + 1$$
$$3g(x) = x^2 + 8$$
$$g(x) = \frac{x^2+8}{3}$$

3 $gf : x \rightarrow x^2 + 1$

$$g[f(x)] = x^2 + 1$$
$$g(x-2) = x^2 + 1$$

Let $x - 2 = u$

$$x = u + 2$$
$$g(u) = (u+2)^2 + 1$$
$$g(u) = u^2 + 4u + 4 + 1$$
$$g(u) = u^2 + 4u + 5$$
$$g(x) = x^2 + 4x + 5$$

4 $g(x) = px + q$

$g^2 : x \rightarrow 49x - 32$

$g^2(x) = 49x - 32$

$g[g(x)] = 49x - 32$

$g(px + q) = 49x - 32$

$p(px + q) + q = 49x - 32$

$p^2x + pq + q = 49x - 32$

By comparison,

$p^2 = 49$ and $pq + q = -32$

$p = 7$ (Given $p > 0$)

When $p = 7$, $7q + q = -32$

$8q = -32$

$q = -4$

5 (a) Let $f^{-1}(x) = y$

$f(y) = x$

$2y - 5 = x$

$y = \dfrac{x+5}{2}$

$f^{-1}(x) = \dfrac{x+5}{2}$

$f^{-1}g(x)$

$= f^{-1}[g(x)]$

$= f^{-1}\left(\dfrac{3x}{x+3}\right)$

$= \dfrac{\left(\dfrac{3x}{x+3}\right)+5}{2}$

$= \dfrac{\dfrac{3x+5(x+3)}{x+3}}{2}$

$= \dfrac{8x+15}{2(x+3)}$

$= \dfrac{8x+15}{2x+6}, x \neq -3$

(b) $g[f(x)]$

$= g(2x-5)$

$= \dfrac{3(2x-5)}{(2x-5)+3}$

$= \dfrac{6x-15}{2x-2}$

$f^2(4) = ff(4)$

$= f(2 \times 4 - 5)$

$= f(3)$

$= 2 \times 3 - 5$

$= 1$

$gf(-k) = f^2(4)$

$\dfrac{6(-k)-15}{2(-k)-2} = 1$

$-6k - 15 = -2k - 2$

$-4k = 13$

$k = -\dfrac{13}{4}$

6 (a) Let $f^{-1}(x) = y$

$f(y) = x$

$p - qy = x$

$y = \dfrac{p-x}{q}$

$f^{-1}(x) = \dfrac{p-x}{q}$

(b) $f(2) = 7$

$p - 2q = -7$

$p = 2q - 7 \ldots$ ①

$f^{-1}(8) = -1$

$\dfrac{p-8}{q} = -1$

$p - 8 = -q$

$p = 8 - q \ldots$ ②

Substitute ① into ② :

$2q - 7 = 8 - q$

$3q = 15$

$q = 5$

From ②: $p = 8 - 5 = 3$

7 (a) Let $f^{-1}(x) = y$

$f(y) = x$

$2y - 3 = x$

$y = \dfrac{x+3}{2}$

$f^{-1}(x) = \dfrac{x+3}{2}$

$f^{-1}g(x) = f^{-1}\left(\dfrac{x}{2}+2\right)$

$= \dfrac{\dfrac{x}{2}+2+3}{2}$

$= \dfrac{x+10}{4}$

(b) $hg(x) = 2x+4$

$h[g(x)] = 2x+4$

$h\left(\dfrac{x}{2}+2\right) = 2x+4$

Let $\dfrac{x}{2}+2 = u$

$\dfrac{x}{2} = u-2$

$x = 2u-4$

$h(u) = 2(u-4)+4$

$h(u) = 2u-8+4$

$h(u) = 2u-4$

$h(x) = 2x-4$

8 (a) $f(y) = ay+b$

$g(y) = \dfrac{5}{3y-b}$

$f(3) = -2$	$g(3) = 1$
$3a+b = -2$	$\dfrac{5}{3(3)-b} = 1$
$3a+4 = -2$	$9-b = 5$
$a = -2$	$b = 4$

(b) The function that maps x onto y is $f^{-1}(x)$.

It is found that $f(y) = -2y+4$.

Let $w = -2y+4$,

$y = \dfrac{4-w}{2}$

$f^{-1}(x) = \dfrac{4-x}{2}$

(c) Hence, the function that maps x onto z is $gf^{-1}(x)$.

$gf^{-1}(x)$

$= g\left(\dfrac{4-x}{2}\right)$

$= \dfrac{5}{3\left(\dfrac{4-x}{2}\right)-4}$

$= \dfrac{10}{3(4-x)-8}$

$= \dfrac{10}{12-3x-8}$

$= \dfrac{10}{4-3x}, \ x \neq \dfrac{4}{3}$

9 (a) $gf(x) = g(h-x^2)$

$= k(h-x^2)+2$

$= hk-kx^2+2$

$= hk+2-kx^2$

But it is given that $gf(x) = 14-3x^2$.

By comparison, $k = 3$

$3h + 2 = 14$

$h = 4$

(b) Let $g^{-1}(-13) = y$

$g(y) = -13$

$3y + 2 = -13$

$y = -5$

$g^{-1}(-13) = -5$

$f(t) = -5$

$4-t^2 = -5$

$t^2 = 9$

$t = \pm 3$

10 (a) $f(x) = \dfrac{hx}{x-3}, x \neq 3$

Let

$f^{-1}(x) = y$.

$f(y) = x$

$\dfrac{hy}{y-3} = x$

$hy = x(y-3)$

$hy = xy-3x$

$xy-hy = 3x$

$y(x-h) = 3x$

$y = \dfrac{3x}{x-h}$

$f^{-1}(x) = \dfrac{3x}{x-h}$

But it is given that

$f^{-1}(x) = \dfrac{kx}{x-2}$.

By comparison,

$k = 3$ and $h = 2$

(b) When $h = 2$, $f^{-1}(x) = \dfrac{3x}{x-2}$

$gf^{-1}(x) = -5x$

$$g\left(\frac{3x}{x-2}\right)=-5x$$

$$\frac{1}{\left(\frac{3x}{x-2}\right)}=-5x$$

$$\frac{x-2}{3x}=-5x$$

$$x-2=-15x^2$$

$$15x^2+x-2=0$$

$$(3x-1)(5x+2)=0$$

$$x=\frac{1}{3} \text{ or } x=-\frac{2}{5}$$

11 (a) $fg(x)=f[g(x)]$

$$=f\left(\frac{2+x}{4-3x}\right)$$

$$=\frac{2\left(\frac{2+x}{4-3x}\right)-1}{\left(\frac{2+x}{4-3x}\right)-3}$$

$$=\frac{\frac{2(2+x)-(4-3x)}{4-3x}}{\frac{2+x-3(4-3x)}{4-3x}}$$

$$=\frac{4+2x-4+3x}{2+x-12+9x}$$

$$=\frac{5x}{10x-10}$$

$$=\frac{x}{2x-2}, \ x\neq 1$$

(b) Let $y=\frac{x}{2x-2}$

$$2xy-2y=x$$

$$2xy-x=2y$$

$$x(2y-1)=2y$$

$$x=\frac{2y}{2y-1}$$

$$(fg)^{-1}=\frac{2x}{2x-1}, \ x\neq\frac{1}{2}$$

12 (a) $fg: x\to x^2+1$

$$f[g(x)]=x^2+1$$

$$2g(x)+2=x^2+1$$

Substitute the x in $f(x)=2x+2$ with $g(x)$.

$$2g(x)=x^2-1$$

$$g(x)=\frac{x^2-1}{2}$$

Hence, $g(3)=\frac{3^2-1}{2}=4$

(b) $f: x\to 2-x$

$$f(x)=2-x$$

Let $f^{-1}(x)=y$

$$f(y)=x$$

$$2-y=x$$

$$y=2-x$$

$$\therefore f^{-1}(x)=2-x$$

$$gf^{-1}: x\to 3x^2-12x+13$$

$$g[f^{-1}(x)]=3x^2-12x+13$$

$$g(2-x)=3x^2-12x+13$$

$$a+b(2-x)^2=3x^2-12x+13$$

$$a+b(4-4x+x^2)=3x^2-12x+13$$

$$a+4b-4bx+bx^2=3x^2-12x+13$$

By comparison,

$b=3$

$$a+4b=13$$

$$a+4(3)=13$$

$$a=1$$

13 (a) $fg: x\to x+2$

$$f[g(x)]=x+2$$

$$a[g(x)]+b=x+2$$

$$a(2x-1)+b=x+2$$

$$2ax-a+b=x+2$$

By comparison,

$2a=1$

$$a=\frac{1}{2}$$

$$-a+b=2$$

$$-\frac{1}{2}+b=2$$

$$b=\frac{5}{2}$$

(b) Let $h^{-1}(x) = y$

$$h(y) = x$$
$$\frac{1}{y-3} = x$$
$$1 = xy - 3x$$
$$xy = 1 + 3x$$
$$y = \frac{1+3x}{x}$$
$$\therefore h^{-1}(x) = \frac{1+3x}{x}$$

$$h^{-1}g(x) = 1$$
$$h^{-1}[g(x)] = 1$$
$$\frac{1+3g(x)}{g(x)} = 1$$
$$1 + 3g(x) = g(x)$$
$$2g(x) = -1$$
$$2(2x-1) = -1$$
$$4x - 2 = -1$$
$$4x = 1$$
$$x = \frac{1}{4}$$

14 (a) $fg(x) = 5x - 3$

$$f[g(x)] = 5x - 3$$
$$g(x) + 14 = 5x - 3$$
$$g(x) = 5x - 17$$

(b) Let $h^{-1}(x) = y$

$$h(y) = x$$
$$\frac{y-1}{y+3} = x$$
$$y - 1 = x(y+3)$$
$$y - 1 = xy + 3x$$
$$y - xy = 3x + 1$$
$$y(1-x) = 3x + 1$$
$$y = \frac{3x+1}{1-x}$$
$$h^{-1}(x) = \frac{3x+1}{1-x}$$

$$g(x) = h^{-1}(x)$$
$$5x - 17 = \frac{3x+1}{1-x}$$
$$(5x-17)(1-x) = 3x + 1$$
$$5x - 5x^2 - 17 + 17x = 3x + 1$$
$$5x^2 - 19x + 18 = 0$$
$$(5x-9)(x-2) = 0$$
$$x = \frac{9}{5} \text{ or } 2$$

15 (a) $g(x) = \frac{2}{x-4}, x \neq k$

Denominator $\neq 0$

$x - 4 \neq 0$

$x \neq 4$

By comparison,

$k = 4$

(b) $f(x) = \frac{5}{2}x - h$

Let

$$f^{-1}(x) = y$$
$$f(y) = x$$
$$\frac{5}{2}y - h = x$$
$$\frac{5}{2}y = x + h$$
$$y = \frac{2(x+h)}{5}$$
$$f^{-1}(x) = \frac{2(x+h)}{5} = \frac{2x+2h}{5}$$

But it is given that

$f^{-1}(x) = \frac{mx+6}{5}$.

By comparison

$m = 2$ and $2h = 6 \Rightarrow h = 3$

(c) When $h = 3$,

$$f(x) = \frac{5}{2}x - 3$$

$$gf(p+1) = p + 2$$
$$g[f(p+1)] = p + 2$$
$$g\left[\frac{5}{2}(p+1) - 3\right] = p + 2$$

Substitute the x in $f(x) = \frac{5}{2}x - 3$ with $(p+1)$.

$$g\left(\frac{5}{2}p + \frac{5}{2} - 3\right) = p + 2$$
$$g\left(\frac{5}{2}p - \frac{1}{2}\right) = p + 2$$
$$\frac{2}{\left(\frac{5}{2}p - \frac{1}{2}\right) - 4} = p + 2$$
$$\frac{2}{\left(\frac{5}{2}p - \frac{9}{2}\right)} = p + 2$$
$$\frac{4}{(5p-9)} = p + 2$$

$4 = (p+2)(5p-9)$

$4 = 5p^2 + p - 18$

$5p^2 + p - 22 = 0$

$(5p+11)(x-2) = 0$

$p = -\frac{11}{5}$ or $p = 2$

16 (a) $f f^{-1}(p^2) = f(p-5)$

$p^2 = -2(p-5)+5$

$p^2 = -2p+10+5$

$p^2 + 2p - 15 = 0$

$(p-3)(p+5) = 0$

$p = 3$ or -5

(b) Let

$f^{-1}(x) = y$

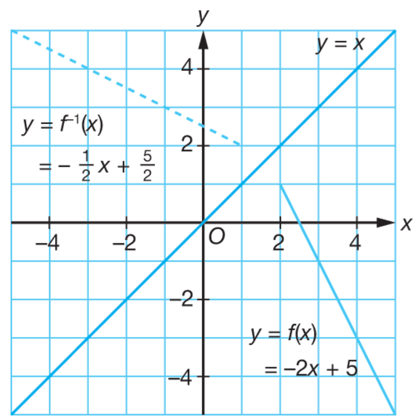
$f(y) = x$

$-2y + 5 = x$

$2y = 5 - x$

$y = \frac{5-x}{2}$

$f^{-1}(x) = -\frac{1}{2}x + \frac{5}{2}$



The graph of f^{-1} is the reflection of the graph of *f* in the straight line *y* = *x*.

(c) The range of *f*(*x*) is $-5 \le x \le 1$.

The domain of $f^{-1}(x)$ is

$-5 \le f^{-1}(x) \le 1$.

The range of $f^{-1}(x)$ is $2 \le f^{-1}(x) \le 5$.

The domain of $f^{-1}(x)$ is the range of *f*(*x*).

The range of $f^{-1}(x)$ is the domain of *f*(*x*).

SPM Spot

1 (a) (i) The function that maps set *B* to set *A* is f^{-1}.

Let $f^{-1}(x) = y$

$f(y) = x$

$2y + 3 = x$

$y = \frac{x-3}{2}$

$\therefore f^{-1}(x) = \frac{x-3}{2}$

(ii) $gf(x) = 6x + 4$

$g[f(x)] = 6x + 4$

$g(2x+3) = 6x + 4$

Let $2x + 3 = u$

$x = \frac{u-3}{2}$

$g(u) = 6\left(\frac{u-3}{2}\right) + 4$

$g(u) = 3(u-3) + 4$

$g(u) = 3u - 5$

$g(x) = 3x - 5$

(b) $|g(x)| < 7$

$|3x-5| < 7$

$-7 < 3x - 5 < 7$

$-7 + 5 < 3x < 7 + 5$

$-2 < 3x < 12$

$-\frac{2}{3} < x < 4$

2 (a) $f(x) = \frac{5}{9}$

$\frac{5}{4x-1} = \frac{5}{9}$

$4x - 1 = 9$

$4x = 10$

$x = \frac{5}{2}$

(b) $f(m+1) = 3fg(p)$

$$\frac{5}{4(m+1)-1} = 3\left(\frac{5}{4p^2+5}\right)$$

$$\frac{1}{4m+3} = \frac{3}{4p^2+3}$$

$$3(4m+3) = 4p^2+3$$

$$12m+9 = 4p^2+3$$

$$12m = 4p^2-6$$

$$6m = 2p^2-3$$

$$m = \frac{2p^2-3}{6}$$

(c) (i) $fg(x) = \frac{3}{4x^2+3}$

$$f[g(x)] = \frac{3}{4x^2+3}$$

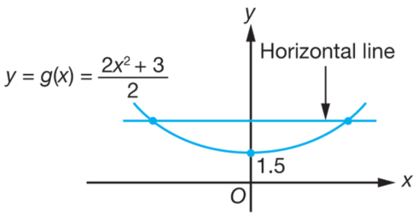
$$\frac{5}{4g(x)-1} = \frac{5}{4x^2+5}$$

$$4g(x)-1 = 4x^2+5$$

$$4g(x) = 4x^2+6$$

$$g(x) = \frac{2x^2+3}{2}$$

(ii)



Since the horizontal line intersects the curve more than one time,

$g(x) = \frac{2x^2+3}{2}$ does not have an inverse function.