Fully-Worked Solutions

FORM 4

CHAPTER 4 Indices, Surds and Logarithms

Self Test 1

1 (a) $\frac{6^{2n} \times 3^{1-n} \times 81}{18^{n-1}} = \frac{(3 \times 2)^{2n} \times 3^{1-n} \times 3^4}{(2 \times 3^2)^{n-1}}$
$= \frac{3^{2n} \times 2^{2n} \times 3^{1-n} \times 3^4}{2^{n-1} \times 3^{2n-2}}$
$= 3^{2n+(1-n)+4-(2n-2)} \times 2^{2n-(n-1)}$
$= 3^{2n+1-n+4-2n+2} \times 2^{2n-n+1}$
$= 3^{7-n} \times 2^{n+1}$

(b) $\frac{5(x^2y)^2 \times 2xy}{(4x^2)^2} = \frac{5x^4y^2 \times 2xy}{16x^4}$
$= \frac{5}{8}x^{4+1-4}y^{2+1}$
$= \frac{5}{8}xy^3$

2 (a) $6^{n+1} + 6^n + 6^{n+2} = 6^n(6^1 + 6^0 + 6^2)$
$= 6^n(6 + 1 + 36)$
$= 6^n(43)$
Thus, $6^{n+1} + 6^n + 6^{n+2}$ is divisible by 43.

(b) $\frac{3 \times 2^{x+1}}{2^{1-x} \times 6^2} = \frac{3 \times 2^{x+1}}{2^{1-x} \times (3 \times 2)^2}$
$= 3^{1-2} \times 2^{x+1-(1-x)-2}$
$= 3^{-1} \times 2^{2x-2}$
$= \frac{2^{2x-2}}{3}$ or $\frac{2^{2x}}{12}$

3 $\frac{2^{m-3} - 2^m}{2^m} = \frac{2^m(2^{-3} - 2^0)}{2^m}$
$= \frac{1}{8} - 1$
$= -\frac{7}{8}$

Thus, $p = -\frac{7}{8}$

Self Test 2

1 $\frac{(4 - \sqrt{3})^2}{2 + \sqrt{3}} = \frac{16 - 8\sqrt{3} + 3}{2 + \sqrt{3}}$
$= \frac{19 - 8\sqrt{3}}{2 + \sqrt{3}} \times \frac{2 - \sqrt{3}}{2 - \sqrt{3}}$
$= \frac{38 - 19\sqrt{3} - 16\sqrt{3} + 24}{(2)^2 - (\sqrt{3})^2}$
$= 62 - 35\sqrt{3}$

2 $\frac{p}{\sqrt{3} - 1} + \frac{1}{\sqrt{3} + 1} = q + 3\sqrt{3}$
$\frac{p(\sqrt{3} + 1) + 1(\sqrt{3} - 1)}{(\sqrt{3} - 1)(\sqrt{3} + 1)} = \frac{(p + 1)\sqrt{3} + (p - 1)}{2}$

Thus, $\frac{p + 1}{2} = 3$ $\frac{p - 1}{2} = q$
$p = 5$ $\frac{5 - 1}{2} = q$
$q = 2$

3 (a) $(2\sqrt{2} - 3)(3\sqrt{2} + 1) = 6(2) + 2\sqrt{2} - 9\sqrt{2} - 3$
$= 12 + (2 - 9)\sqrt{2} - 3$
$= 9 - 7\sqrt{2}$

(b) $\frac{\sqrt{20} - \sqrt{100}}{\sqrt{625} - \sqrt{125}} = \frac{\sqrt{4 \times 5} - 10}{25 - \sqrt{25 \times 5}}$
$= \frac{2\sqrt{5} - 10}{25 - 5\sqrt{5}}$
$= \frac{2(\sqrt{5} - 5)}{5(5 - \sqrt{5})}$
$= \frac{-2(5 - \sqrt{5})}{5(5 - \sqrt{5})}$
$= -\frac{2}{5}$

4 (a) Area of triangle $= \frac{1}{2}(AB)(BC)$
$\frac{47}{2} = \frac{1}{2}(4\sqrt{3} + 1)(BC)$
$BC = \frac{47}{4\sqrt{3} + 1} \times \frac{4\sqrt{3} - 1}{4\sqrt{3} - 1}$
$= \frac{188\sqrt{3} - 47}{16(3) - 1}$
$= \frac{47(4\sqrt{3} - 1)}{47}$
$= (4\sqrt{3} - 1)$ cm

(b) $AC = \sqrt{(4\sqrt{3} + 1)^2 + (4\sqrt{3} - 1)^2}$
$= \sqrt{16(3) + 1 + 8\sqrt{3} + 16(3) - 8\sqrt{3} + 1}$
$= \sqrt{48 + 1 + 48 + 1}$
$= \sqrt{98}$
$= \sqrt{49(2)}$
$= 7\sqrt{2}$ cm

Self Test 3

1 (a) $\log_2 (pq^2) = -4$
$pq^2 = 2^{-4}$
$pq^2 = \frac{1}{16}$
$q^2 = \frac{1}{16p}$ .... ①
$\log_2 \frac{p}{q^2} = 3$
$\frac{p}{q^2} = 2^3$
$p = 8q^2$ .... ②

Substitute ① into ②,

$$p = 8\left(\frac{1}{16p}\right)$$
$$p^2 = \frac{1}{2}$$
$$p^2 = 2^{-1}$$
$$p = 2^{-\frac{1}{2}}$$

From ①,

$$q^2 = \frac{1}{16p}$$
$$= \frac{1}{2^4 \times 2^{-\frac{1}{2}}}$$
$$= \frac{1}{2^{\frac{7}{2}}}$$
$$= 2^{-\frac{7}{2}}$$
$$q = 2^{-\frac{7}{4}}$$

(b) $\log_2(pq) = \log_2 p + \log_2 q$

$$= \log_2\left(2^{-\frac{1}{2}}\right) + \log_2\left(2^{-\frac{7}{4}}\right)$$
$$= -\frac{1}{2} + \left(-\frac{7}{4}\right)$$
$$= -\frac{9}{4}$$

2 $\log_3 p = a$

$$\frac{\log_9 p}{\log_9 3} = a$$
$$\log_9 p = a\left(\log_9 9^{\frac{1}{2}}\right)$$
$$\log_9 p = \frac{1}{2}a$$
$$\log_{27} q = b$$
$$\frac{\log_9 q}{\log_9 27} = b$$
$$\log_9 q = b\left(\frac{\log_3 27}{\log_3 9}\right)$$
$$\log_9 q = b\left(\frac{3}{2}\right)$$
$$\log_9 q = \frac{3}{2}b$$
$$\log_9 \sqrt{p^4q^3} = \log_9 (p^4q^3)^{\frac{1}{2}}$$
$$= \frac{1}{2}\log_9 p^4 + \frac{1}{2}\log_9 q^3$$
$$= \frac{1}{2}(4)\log_9 p + \frac{1}{2}(3)\log_9 q$$
$$= 2\left(\frac{1}{2}a\right) + \frac{3}{2}\left(\frac{3}{2}b\right)$$
$$= a + \frac{9}{4}b$$

3 $\log_2 3 = 1.585$, $\log_2 5 = 2.322$

$$\log_{15}\frac{50}{3} = \frac{\log_2\left(\frac{50}{3}\right)}{\log_2 15}$$
$$= \frac{\log_2 50 - \log_2 3}{\log_2 15}$$
$$= \frac{\log_2(5^2 \times 2) - \log_2 3}{\log_2(3 \times 5)}$$
$$= \frac{\log_2 5^2 + \log_2 2 - \log_2 3}{\log_2 3 + \log_2 5}$$
$$= \frac{2(2.322) + 1 - 1.585}{1.585 + 2.322}$$
$$= \frac{4.059}{3.907}$$
$$= 1.039$$

Self Test 4

1
$$8^{q-1} \times 2^{2p+1} = 4^7$$
$$2^{3q-3} \times 2^{2p+1} = 2^{14}$$
$$3q - 3 + 2p + 1 = 14$$
$$2p + 3q = 16 \text{ .... ①}$$
$$9^{p-4} \times 3^q = 81$$
$$3^{2p-8} \times 3^q = 3^4$$
$$2p - 8 + q = 4$$
$$2p + q = 12 \text{ .... ②}$$
①−②: $2q = 4$

$$q = 2$$

From ②,

$$2p + 2 = 12$$
$$2p = 10$$
$$p = 5$$

2
$$5^{2x+1} - 5^{x+1} + 2 = 2(5^x)$$
$$5^{2x} \times 5 - 5^x \times 5 + 2 = 2(5^x)$$

Let $y = 5^x$

$$5y^2 - 5y + 2 = 2y$$
$$5y^2 - 7y + 2 = 0$$
$$(5y - 2)(y - 1) = 0$$
$$y = \frac{2}{5}, y = 1$$

Thus,

$$5^x = \frac{2}{5}$$
$$\lg 5^x = \lg\left(\frac{2}{5}\right)$$
$$x = -\frac{0.398}{0.699}$$
$$= -0.569$$

$$5^x = 1$$
$$5^x = 5^0$$
$$x = 0$$

3 (a) $\log_5(5^{x-1} + 100) = 3$

$$5^{x-1} + 100 = 5^3$$
$$5^{x-1} = 25$$
$$5^{x-1} = 5^2$$
$$x - 1 = 2$$
$$x = 3$$

(b) $\log_2(3x + 3) = 2 + \log_2 3x$

$$\log_2(3x + 3) - \log_2 3x = 2$$
$$\log_2\left(\frac{3x + 3}{3x}\right) = 2$$
$$\frac{3x + 3}{3x} = 2^2$$
$$3x + 3 = 12x$$
$$9x = 3$$
$$x = \frac{1}{3}$$

(c) $3^{p+1} = \frac{7}{10}$

$$\lg 3^{p+1} = \lg 0.7$$
$$p + 1 = \frac{\lg 0.7}{\lg 3}$$
$$p + 1 = -0.325$$
$$p = -1.325$$

4 (a) $\sqrt{5x - 6} - \sqrt{2} = 2$

$$\left(\sqrt{5x - 6}\right)^2 = \left(2 + \sqrt{2}\right)^2$$
$$5x - 6 = 4 + 4\sqrt{2} + 2$$
$$5x - 6 = 6 + 4\sqrt{2}$$
$$5x = 12 + 4\sqrt{2}$$
$$x = \frac{12 + 4\sqrt{2}}{5}$$

(b) $\left(2+3\sqrt{2}\right)x-5\sqrt{2}+1=\left(1+\sqrt{2}\right)x$

$$\left(2+3\sqrt{2}\right)x-\left(1+\sqrt{2}\right)x=5\sqrt{2}-1$$
$$\left(2+3\sqrt{2}-1-\sqrt{2}\right)x=5\sqrt{2}-1$$
$$\left(1+2\sqrt{2}\right)x=5\sqrt{2}-1$$
$$x=\frac{5\sqrt{2}-1}{1+2\sqrt{2}}\times\frac{1-2\sqrt{2}}{1-2\sqrt{2}}$$
$$=\frac{5\sqrt{2}-10(2)-1+2\sqrt{2}}{1-(2^2\times 2)}$$
$$=\frac{7\sqrt{2}-21}{-7}$$
$$=3-\sqrt{2}$$

SPM Practice

Paper 1

1 $\dfrac{5^{x+7}\times 625^{2x-1}}{125^{2-x}}=\dfrac{5^{x+7}\times(5^4)^{2x-1}}{(5^3)^{2-x}}$
$$=\frac{5^{x+7}\times 5^{8x-4}}{5^{6-3x}}$$
$$=5^{x+7+(8x-4)-(6-3x)}$$
$$=5^{x+7+8x-4-6+3x}$$
$$=5^{12x-3}$$

2 $2^{2x+3}\times(2\times 3)^{x-5}\times(2^3)^{-\frac{2}{3}x-1}=p^{ax+b}$
$$2^{2x+3}\times 2^{x-5}\times 3^{x-5}\times 2^{-2x-3}=p^{ax+b}$$
$$2^{2x+3+x-5+(-2x-3)}\times 3^{x-5}=p^{ax+b}$$
$$2^{x-5}\times 3^{x-5}=p^{ax+b}$$
$$6^{x-5}=p^{ax+b}$$
$\therefore p=6, a=1, b=-5$

3 $y=2^x$
$$2^{x+1}-8^{x+2}+4^{x-1}=2^{x+1}-(2^3)^{x+2}+(2^2)^{x-1}$$
$$=2^x\times 2-(2^x)^3\times 2^6+(2^x)^2\times 2^{-1}$$
$$=2y-64y^3+\frac{1}{2}y^2$$

4 (a) $\log_6\left(\frac{4}{15}\times\frac{32}{7}\times\left(\frac{3}{2}\right)^2\div\frac{8}{105}\right)$
$$=\log_6\left(\frac{4}{15}\times\frac{32}{7}\times\frac{9}{4}\times\frac{105}{8}\right)$$
$$=\log_6 36$$
$$=\log_6 6^2$$
$$=2$$

(b) $3\log_2 32-4\log_3\frac{1}{9}$
$$=3\log_2 2^5-4\log_3 3^{-2}$$
$$=3(5)-4(-2)$$
$$=15+8$$
$$=23$$

5 $2+\log_3 2y-2\log_3 x=x$
$$\log_3 2y-\log_3 x^2=x-2$$
$$\log_3\left(\frac{2y}{x^2}\right)=x-2$$
$$\frac{2y}{x^2}=3^{x-2}$$
$$2y=(3^{x-2})x^2$$
$$y=\frac{(3^{x-2})x^2}{2}$$

6 $\log_a x=p, \log_a y=q$

(a) $\log_a x^3y^2=\log_a x^3+\log_a y^2$
$$=3\log_a x+2\log_a y$$
$$=3p+2q$$

(b) $\log_x(xy)^{\frac{1}{2}}=\frac{1}{2}\left[\frac{\log_a(xy)}{\log_a x}\right]$
$$=\frac{1}{2}\left[\frac{\log_a x+\log_a y}{\log_a x}\right]$$
$$=\frac{1}{2}\left(\frac{p+q}{p}\right)$$
$$=\frac{1}{2}+\frac{q}{2p}$$

7 (a) $7^{3x}\times 343^{1+x}=\frac{1}{49^{2x}}$
$$7^{3x}\times 7^{3+3x}=7^{-4x}$$
$$7^{3x+3+3x}=7^{-4x}$$
$$6x+3=-4x$$
$$10x=-3$$
$$x=-\frac{3}{10}$$

(b) $3^{x+1}=6^{1-3x}$
$$\lg 3^{x+1}=\lg 6^{1-3x}$$
$$(x+1)\lg 3=(1-3x)\lg 6$$
$$0.477x+0.477=0.778-2.334x$$
$$2.811x=0.301$$
$$x=0.107$$

8 (a) $3^x+3^{x+2}=90$

Let $y=3^x$
$$y+y(3^2)=90$$
$$y+9y=90$$
$$10y=90$$
$$y=9$$
$$3^x=9$$
$$3^x=3^2$$
$\therefore x=2$

(b) $2\lg(x+2)+\lg 4=\lg x+4\lg 3$
$$\lg(x+2)^2+\lg 4=\lg x+\lg 3^4$$
$$\lg 4(x+2)^2=\lg(81x)$$
$$4(x+2)^2=(81x)$$
$$4(x^2+4x+4)=81x$$
$$4x^2-65x+16=0$$
$$(4x-1)(x-16)=0$$
$$x=\frac{1}{4}, x=16$$

9 $\left(3\sqrt{5}-2\sqrt{2}\right)^2=\left(3\sqrt{5}-2\sqrt{2}\right)\left(3\sqrt{5}-2\sqrt{2}\right)$
$$=9(5)-6\sqrt{10}-6\sqrt{10}+4(2)$$
$$=45-12\sqrt{10}+8$$
$$=53-12\sqrt{10}$$

Thus, $3\sqrt{3}-2\sqrt{2}$ is the square root of $53-12\sqrt{10}$.

10 $\dfrac{\sqrt{8}}{\sqrt{7}-\sqrt{5}}\times\dfrac{\sqrt{7}+\sqrt{5}}{\sqrt{7}+\sqrt{5}}=\dfrac{\sqrt{56}+\sqrt{40}}{7-5}$
$$=\frac{\sqrt{4\times 14}+\sqrt{4\times 10}}{2}$$
$$=\frac{2\sqrt{14}+2\sqrt{10}}{2}$$
$$=\sqrt{14}+\sqrt{10}$$

Paper 2

1 (a) $m = 2^x$ $\quad$ $n = 2^y$

$\log_2 m = \log_2 2^x$ $\quad$ $\log_2 n = \log_2 2^y$

$\log_2 m = x$ $\quad$ $\log_2 n = y$

$$\log_2\left(\frac{\sqrt{m^3 n}}{64}\right) = \log_2 (m^3 n)^{\frac{1}{2}} - \log_2 64$$
$$= \frac{1}{2}(\log_2 m^3 + \log_2 n) - \log_2 2^6$$
$$= \frac{1}{2}(3\log_2 m + \log_2 n) - 6$$
$$= \frac{1}{2}(3x + y) - 6$$
$$= \frac{3x + y - 12}{2}$$

(b) $p = 2^{2x}$

(i) $2^{2x} + 4^{x+2} = 2^{2x} + (2^2)^{x+2}$
$= 2^{2x} + 2^{2x+4}$
$= p + p(2^4)$
$= p + 16p$
$= 17p$

(ii) $2^{2x-1} - 4^{2x+1} + 16^{x-1}$
$= 2^{2x} \times 2^{-1} - (2^2)^{2x+1} + (2^4)^{x-1}$
$= \frac{2^{2x}}{2} - 2^{4x+2} + 2^{4x-4}$
$= \frac{2^{2x}}{2} - (2^{2x})^2 \times 2^2 + (2^{2x})^2 \times 2^{-4}$
$= \frac{p}{2} - 4p^2 + p^2\left(\frac{1}{16}\right)$
$= \frac{8p - 64p^2 + p^2}{16}$
$= \frac{8p - 63p^2}{16}$

2 (a) (i) $\log_3 N + \log_9 N = 6$

$\log_3 N + \frac{\log_3 N}{\log_3 9} = 6$

$\log_3 N + \frac{\log_3 N}{2} = 6$

$\log_3 N^2 + \log_3 N = 12$

$\log_3 (N^3) = 12$

$N^3 = 3^{12}$

$N^3 = (3^4)^3$

$N = 3^4$

$N = 81$

(ii) $\log_4 x = 9\log_x 4$

$\frac{\log_x x}{\log_x 4} = 9\log_x 4$

$1 = 9(\log_x 4)^2$

$(\log_x 4)^2 = \frac{1}{9}$

$\log_x 4 = \frac{1}{3}$, $\quad$ $\log_x 4 = -\frac{1}{3}$

$x = 4^{\frac{1}{3}}$ $\quad$ $x = 4^{-\frac{1}{3}}$

(b) (i) $\left(\frac{1}{4}\right)^{-2} \div \left(\frac{1}{576}\right)^{-\frac{1}{2}} \times 243^{\frac{3}{5}}$

$= (2^{-2})^{-2} \div (24^{-2})^{-\frac{1}{2}} \times (3^5)^{\frac{3}{5}}$
$= 2^4 \div 24 \times 3^3$
$= \frac{16}{24} \times 27$
$= 18$

(ii) $32^{x-1} \times 4^{3x-2} \div \frac{1}{8^{4x+3}}$

$= (2^5)^{x-1} \times (2^2)^{3x-2} \div \frac{1}{(2^3)^{4x+3}}$
$= 2^{5x-5} \times 2^{6x-4} \div 2^{-12x-9}$
$= 2^{5x-5+(6x-4)-(-12x-9)}$
$= 2^{5x-5+6x-4+12x+9}$
$= 2^{23x}$

3 (a) $\log_7 p = x$, $\log_7 q = y$

(i) $\log_{49}\frac{pq^3}{7} = \frac{\log_7 pq^3}{\log_7 49}$
$= \frac{\log_7 p + \log_7 q^3}{2}$
$= \frac{x + 3y}{2}$

(ii) $\log_{\frac{1}{7}}\sqrt{\frac{p}{q}} = \frac{\log_7\left(\frac{p}{q}\right)^{\frac{1}{2}}}{\log_7 \frac{1}{7}}$
$= \frac{\frac{1}{2}(\log_7 p - \log_7 q)}{\log_7 7^{-1}}$
$= -\frac{1}{2}(x - y)$

(b) Area of trapezium $= \frac{1}{2}(x + 2 + 4 + \sqrt{5})(\sqrt{49 - 4})$
$= \frac{1}{2}(x + 6 + \sqrt{5})(\sqrt{45})$
$= \frac{3}{2}(x + 6 + \sqrt{5})(\sqrt{5})$
$= \frac{3}{2}(\sqrt{5}x + 6\sqrt{5} + 5)$
$= \frac{3\sqrt{5}}{2}x + 9\sqrt{5} + \frac{15}{2}$

4 (a) $9^x + 3 = 4(3^x)$

Let $y = 3^x$

$9^x + 3 = 4(3^x)$

$(3^x)^2 + 3 = 4(3^x)$

$y^2 + 3 = 4y$

$y^2 - 4y + 3 = 0$

$(y - 1)(y - 3) = 0$

$y = 1$, $y = 3$

$3^x = 1 = 3^0$, $\quad$ $3^x = 3^1$

$\therefore x = 0, x = 1$

(b) $2 + \lg(2x - 3) = \lg x^2 + \lg 25$

$\lg 100 + \lg(2x - 3) = \lg x^2 + \lg 25$

$\lg(100)(2x - 3) = \lg 25x^2$

$200x - 300 = 25x^2$

$8x - 12 = x^2$

$x^2 - 8x + 12 = 0$

$(x - 2)(x - 6) = 0$

$x = 2, x = 6$

5 (a) (i) $y = 275(1.05)x$

$y = 10.2$, $275(1.05)^x = 10.2$

$1.05^x = 0.0371$

$\lg(1.05^x) = \lg(0.0371)$

$x(0.02119) = -1.4306$

$x = -67.51$

(ii) $x = 7.5$, $y = 275(1.05)^{7.5} = 396.51$

(b) Total amount $= 500\,(1.0375)^{t+1} > 5\,000$

$$(1.0375)^{t+1} > 10$$
$$\lg\,(1.0375)^{t+1} > \lg 10$$
$$(t+1)(0.016) > 1$$
$$t+1 > 62.5$$
$$t > 61.5$$

$\therefore$ May saved her money into the account for 62 years.

(c)
$$\frac{a \times (3a^2)^2}{\sqrt{27a^2}} = \frac{a\left(9a^4\right)}{\sqrt{9 \times 3a^2}}$$
$$= \frac{9a^5}{3a\sqrt{3}}$$
$$= \frac{9a^4}{3\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$
$$= \frac{9\sqrt{3}\,a^4}{3(3)}$$
$$= \sqrt{3}\,a^4$$
$$= 3^{\frac{1}{2}}a^4$$