FORM 5

CHAPTER 3 Integration

Self Test 1

1 $\frac{d}{dx}\left[(1+2x)^{\frac{3}{2}}\right] = 3(1+2x)^{\frac{1}{2}}$

$\int 3(1+2x)^{\frac{1}{2}}\,dx = 3 \times \frac{1}{3}(1+2x)^{\frac{3}{2}}$

$= \frac{1}{3}(1+2x)^{\frac{3}{2}}$

2 $f'(x) = \frac{2}{3}x - 5$ .... ①

$f(x) = ax^2 - bx + 3$

$f'(x) = 2ax - b$ .... ②

Compare ① with ②,

$\therefore b = 5$

$2a = \frac{2}{3}$

$a = \frac{1}{3}$

$\int\left(\frac{2}{3}x - 5\right)dx = \frac{1}{3}x^2 - 5x + 3$

$3\int\left(\frac{2}{3}x - 5\right)dx = 3\left(\frac{1}{3}x^2 - 5x + 3\right)$

$\int(2x - 15)\,dx = x^2 - 15x + 9$

3 $y = \frac{2}{3}x^3 + x - \frac{4}{5}$

$\frac{dy}{dx} = 2x^2 + 1$

$\int(8x^2 + 4)\,dx = 4\int(2x^2 + 1)\,dx$

$= 4\left[\frac{2}{3}x^3 + x - \frac{4}{5}\right]$

$= \frac{8}{3}x^2 + 4x - \frac{16}{5}$

Self Test 2

1 $\frac{dy}{dx} = 3x^2 + 4x$

$\int(3x^2 + 4x)\,dx = \frac{3x^3}{3} + \frac{4x^2}{2} + c$

$y = x^3 + 2x^2 + c$

$(-2, 4),\ 4 = (-2)^3 + 2(-2)^2 + c$

$4 = -8 + 8 + c$

$c = 4$

$\therefore y = x^3 + 2x^2 + 4$

2 (a) $\int\sqrt[3]{3x-1}\,dx = \int(3x-1)^{\frac{1}{3}}\,dx$

$= \frac{(3x-1)^{\frac{4}{3}}}{\frac{4}{3}(3)} + c$

$= \frac{1}{4}(3x-1)^{\frac{4}{3}} + c$

(b) $\int\frac{2}{5}(6-5x)^{-3} + 4x^2 - 3\,dx$

$= \frac{2}{5} \times \frac{(6-5x)^{-2}}{-2(-5)} + \frac{4x^3}{3} - 3x + c$

$= \frac{(6-5x)^{-2}}{25} + \frac{4x^3}{3} - 3x + c$

$= \frac{1}{25(6-5x)^2} + \frac{4x^3}{3} - 3x + c$

(c) $\int x(3-x^2)^2\,dx$

$= \int x(9 - 6x^2 + x^4)\,dx$

$= \int 9x - 6x^3 + x^5\,dx$

$= \frac{9}{2}x^2 - \frac{3}{2}x^4 + \frac{1}{6}x^6 + c$

3 $\frac{dy}{dx} = 3x^2 - 2x + 3$

$\int 3x^2 - 2x + 3\,dx = x^3 - x^2 + 3x + c$

$f(x) = x^3 - x^2 + 3x + c$

x-intercept $= 4$, $\quad (x, y) = (4, 0)$

$4^3 - 4^2 + 3(4) + c = 0$

$64 - 16 + 12 + c = 0$

$c = -60$

$\therefore f(x) = x^3 - x^2 + 3x - 60$

Self Test 3

1 $\int_k^2 (2x+1)^{-2}\,dx = \frac{1}{5}$

$\left[\frac{(2x+1)^{-1}}{(-1)(2)}\right]_k^2 = \frac{1}{5}$

$\left(\frac{5^{-1}}{-2}\right) - \left[\frac{(2k+1)^{-1}}{-2}\right] = \frac{1}{5}$

$-\frac{1}{10} + \frac{1}{2(2k+1)} = \frac{1}{5}$

$\frac{1}{2(2k+1)} = \frac{3}{10}$

$2k + 1 = \frac{5}{3}$

$2k = \frac{2}{3}$

$k = \frac{1}{3}$

2 (a) $y = (x-2)^{\frac{1}{3}}$

$y^3 = x - 2$

$x = y^3 + 2$

Area $= \int_1^2 y^3 + 2\,dy$

$= \left[\frac{y^4}{4} + 2y\right]_1^2$

$= \left[\frac{2^4}{4} + 2(2)\right] - \left[\frac{1^4}{4} + 2(1)\right]$

$= 8 - \frac{9}{4}$

$= \frac{23}{4}$ units^2

(b) Area $= \int_1^2 x^2 + 1\, dx$

$= \left[\frac{x^3}{3} + x\right]_1^2$

$= \left[\frac{2^3}{3} + 2\right] - \left[\frac{1^3}{3} + 1\right]$

$= \frac{10}{3}$ units^2

3 (a) $y = 5 - 2x$ ...... ①

$y = x^2 + 2$ .... ②

① = ②:

$x^2 + 2 = 5 - 2x$

$x^2 + 2x - 3 = 0$

$(x + 3)(x - 1) = 0$

$x = -3, \quad x = 1$

$x = -3, \quad y = 5 - 2(-3) = 11$

$x = 1, \quad y = 5 - 2(1) = 3$

$\therefore A(-3, 11), B(1, 3)$

(b) Area of the shaded region

$= \int_{-3}^{1} 5 - 2x\; dx - \int_{-3}^{1} x^2 + 2\, dx$

$= \int_{-3}^{1} (5 - 2x) - (x^2 + 2)\, dx$

$= \int_{-3}^{1} 3 - 2x - x^2\; dx$

$= \left[3x - x^2 - \frac{x^3}{3}\right]_{-3}^{1}$

$= \left[3 - 1 - \frac{1}{3}\right] - \left[-9 - 9 - \frac{(-27)}{3}\right]$

$= \frac{5}{3} - (-9)$

$= \frac{32}{3}$ units^2

4 (a) Volume $= \pi \int y^2\, dx$

$= \pi \int_2^4 x - 1\, dx$

$= \pi \left[\frac{x^2}{2} - x\right]_2^4$

$= \pi \left[\left(\frac{16}{2} - 4\right) - \left(\frac{4}{2} - 2\right)\right]$

$= 4\pi$ units^3

(b) $y = \frac{1}{\sqrt{x + 2}}$

$\sqrt{x + 2} = \frac{1}{y}$

$x + 2 = \frac{1}{y^2}$

$x = \frac{1}{y^2} - 2$

Volume $= \pi \int_1^2 \left(\frac{1}{y^2} - 2\right)^2 dy$

$= \pi \int_1^2 \left(\frac{1}{y^4} - \frac{4}{y^2} + 4\right) dy$

$= \pi \int_1^2 \left(y^{-4} - 4y^{-2} + 4\right) dy$

$= \pi \left[\frac{y^{-3}}{-3} - \frac{4y^{-1}}{-1} + 4y\right]_1^2$

$= \pi \left[-\frac{1}{3y^3} + \frac{4}{y} + 4y\right]_1^2$

$= \pi \left[\left(-\frac{1}{3(8)} + \frac{4}{2} + 4(2)\right) - \left(-\frac{1}{3} + 4 + 4\right)\right]$

$= \pi \left(\frac{239}{24} - \frac{23}{3}\right)$

$= \frac{55}{24}\pi$ units^3

5 Volume $= \frac{8}{5}\pi$

$\pi \int_p^0 \left(\frac{1}{2}x^2\right)^2 dx = \frac{8}{5}\pi$

$\pi \int_p^0 \frac{1}{4}x^4\, dx = \frac{8}{5}\pi$

$\frac{1}{4}\left[\frac{x^5}{5}\right]_p^0 = \frac{8}{5}$

$-\frac{p^5}{5} = \frac{32}{5}$

$p^5 = -32$

$= (-2)^5$

$p = -2$

6 (a) $P = y$-intercept, $x = 0$

$y = 2(0) + 3 = 3$

$\therefore P(0, 3)$

$6 - x = 2x^2 + 3$

$2x^2 + x - 3 = 0$

$(2x + 3)(x - 1) = 0$

$x = -\frac{3}{2}, x = 1$

$x = 1, y = 6 - 1 = 5$

$\therefore Q(1, 5)$

(b) Area $= \int_0^1 (6 - x) - (2x^2 + 3)\, dx$

$= \int_0^1 3 - x - 2x^2\, dx$

$= \left[3x - \frac{x^2}{2} - \frac{2x^3}{3}\right]_0^1$

$= 3 - \frac{1}{2} - \frac{2}{3} - 0$

$= \frac{11}{6}$ units^2

(c) Volume $= \pi \int_3^5 (2x^2 + 3)^2\, dx + \pi \int_5^6 (6 - x)^2\, dx$

$= \pi \int_3^5 4x^4 + 12x^2 + 9\, dx + \pi \int_5^6 (6 - x)^2\, dx$

$= \pi \left[\frac{4x^5}{5} + \frac{12x^3}{3} + 9x\right]_3^5 + \pi \left[\frac{(6 - x)^3}{-3}\right]_5^6$

$= \pi \left[\left(\frac{4}{5}(3\,125) + 4(5^3) + 45\right) - \left(\frac{4}{5}(243) + 4(3^3) + 27\right)\right]$

$- \frac{1}{3}\pi(0 - 1)$

$= 2\,715.6\pi + \frac{1}{3}\pi$

$= 2\,715.93\pi$ units^3

Self Test 4

1 (a) $m_1 = 4x - 5$

$x = 2, m_1 = 4(2) - 5 = 3$

Since the two tangents are perpendicular at (2, 3), then

$m_2 = -\frac{1}{3}$

$m_2 = px - 3$

$-\frac{1}{3} = p(2) - 3$

$-\frac{1}{3} = 2p - 3$

$p = \frac{4}{3}$

(b) $m_1 \times m_2 = -1$

$$(4x-5)\left(\frac{4}{3}x-3\right) = -1$$
$$\frac{16}{3}x^2 - 12x - \frac{20}{3}x + 15 = -1$$
$$16x^2 - 36x - 20x + 45 = -3$$
$$16x^2 - 56x + 48 = 0$$
$$2x^2 - 7x + 6 = 0$$
$$(2x-3)(x-2) = 0$$
$$x = \frac{3}{2},\ x = 2$$

$\therefore$ x-coordinate $= \frac{3}{2}$

(c) $y_1 = \int 4x - 5\,dx$
$$= \frac{4x^2}{2} - 5x + c$$
$$y_1 = 2x^2 - 5x + c$$
$$(2, 3),\ 3 = 2(4) - 5(2) + c$$
$$c = 5$$
$$\therefore y_1 = 2x^2 - 5x + 5$$

$$y_2 = \int \frac{4}{3}x - 3\,dx$$
$$= \frac{2}{3}x^2 - 3x + c$$
$$(2, 3), 3 = \frac{2}{3}(2)^2 - 3(2) + c$$
$$c = \frac{19}{3}$$
$$\therefore y_2 = \frac{2}{3}x^2 - 3x + \frac{19}{3}$$

2 $\frac{dy}{dx} = 3x^2 + 4$
$$y = \int 3x^2 + 4\,dx$$
$$= x^3 + 4x + c$$
$$(-1, 1),\ 1 = (-1)^3 + 4(-1) + c$$
$$c = 6$$
$$\therefore y = x^3 + 4x + 6$$
At $x = -2$, $y = (-2)^3 + 4(-2) + 6$
$$= -10$$
Equation of tangent: $\frac{y+10}{x+2} = 3(-2)^2 + 4$
$$y + 10 = 16(x+2)$$
$$y = 16x + 22$$

3 Volume $= \pi\int_0^{12} y + 3\,dy$
$$= \pi\left[\frac{y^2}{2} + 3y\right]_0^{12}$$
$$= \pi\left[\frac{144}{2} + 3(12) - 0\right]$$
$$= 108\pi \text{ units}^3$$

SPM Practice

Paper 1

1 $y = (5-2x)^3$
$$\frac{dy}{dx} = 3(5-2x)^2(-2)$$
$$= -6(5-2x)^2 \text{ (Shown)}$$
$$\int -6(5-2x)^2 dx = (5-2x)^3$$
$$\therefore \int (5-2x)^2 dx = -\frac{(5-2x)^3}{6}$$

2 (a) $\int (5x-1)^{-\frac{1}{2}}\,dx = \frac{(5x-1)^{\frac{1}{2}}}{\frac{1}{2}(5)} + c$
$$y = \frac{2}{5}\sqrt{5x-1} + c$$
$$(1, -3),\ -3 = \frac{2}{5}\sqrt{5(1)-1} + c$$
$$-3 = \frac{4}{5} + c$$
$$c = -\frac{19}{5}$$
$$\therefore y = \frac{2}{5}\sqrt{5x-1} - \frac{19}{5}$$

(b) $\frac{dy}{dx} = x - \frac{a}{x^2}$

(i) $y = -3x - 1 \Rightarrow m = -3$

At $(-2, 5)$, $\frac{dy}{dx} = -2 - \frac{a}{(-2)^2}$
$$-3 = -2 - \frac{a}{4}$$
$$\frac{a}{4} = 1$$
$$a = 4$$

(ii) $y = \int x - \frac{4}{x^2}\,dx$
$$= \int x - 4x^{-2}\,dx$$
$$= \frac{x^2}{2} + \frac{4}{x} + c$$
$$(-2, 5),\ 5 = \frac{(-2)^2}{2} + \frac{4}{(-2)} + c$$
$$5 = 2 - 2 + c$$
$$c = 5$$
$$\therefore y = \frac{x^2}{2} + \frac{4}{x} + 5$$

3 $\frac{dy}{dx} = kx^3$

$x = 2$, $\frac{dy}{dx} = 16$
$$k(2)^3 = 16$$
$$8k = 16$$
$$k = 2$$
$$y = \int 2x^3\,dx$$
$$= \frac{x^4}{2} + c$$
$$x = 2, y = 12, 12 = \frac{16}{2} + c$$
$$c = 4$$
$$\therefore y = \frac{x^4}{2} + 4$$
When $x = 1$, $y = \frac{1^4}{2} + 4 = \frac{9}{2}$

4 (a) $\int_1^4 f(x)\,dx = 10$

(i) $\int_1^4 \left[2f(x) - \sqrt{x}\right] dx$
$$= \int_1^4 2f(x)\,dx - \int_1^4 x^{\frac{1}{2}}\,dx$$
$$= 20 - \left[\frac{2}{3}x^{\frac{3}{2}}\right]_1^4$$
$$= 20 - \left[\frac{2}{3}(8) - \frac{2}{3}\right]$$
$$= 20 - \frac{14}{3}$$
$$= \frac{46}{3}$$

(ii) $\int_4^1 3 - 4f(x)\,dx$

$= -\int_1^4 3 - 4f(x)\,dx$

$= \int_1^4 4f(x) - 3\,dx$

$= 4\int_1^4 f(x)\,dx - \int_1^4 3\,dx$

$= 4(10) - [3x]_1^4$

$= 40 - [12 - 3]$

$= 31$

(iii) $\int_1^4 k - f(x)\,dx = -8$

$\int_1^4 k\,dx - \int_1^4 f(x)\,dx = -8$

$[kx]_1^4 - 10 = -8$

$4k - k = 2$

$3k = 2$

$k = \frac{2}{3}$

(b) $\int_{-1}^2 \frac{2x^3 + r}{3}\,dx = -\frac{1}{6}$

$\int_{-1}^2 2x^3 + r\,dx = -\frac{1}{2}$

$\left[\frac{1}{2}x^4 + rx\right]_{-1}^2 = -\frac{1}{2}$

$\left[\frac{2(16)}{4} + 2r\right] - \left[\frac{2}{4} - r\right] = -\frac{1}{2}$

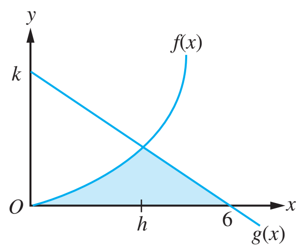
$8 + 2r - \frac{1}{2} + r = -\frac{1}{2}$

$3r = -8$

$r = -\frac{8}{3}$

5 $\int_0^h f(x)\,dx + \int_h^6 g(x)\,dx = 20$

(a)



(b) $\int_0^6 g(x)\,dx$ = Area of triangle

$= \frac{1}{2}k(6)$

$= 3k$ units^2

6 $\frac{d}{dx}\left(x\sqrt{x-1}\right) = 1\sqrt{x-1} + x\left(\frac{1}{2}\right)(x-1)^{-\frac{1}{2}}$

$= \sqrt{x-1} + \frac{x}{2\sqrt{x-1}}$

$= \frac{2(x-1) + x}{2\sqrt{x-1}}$

$= \frac{2x - 2 + x}{2\sqrt{x-1}}$

$= \frac{3x - 2}{2\sqrt{x-1}}$ (Shown)

$\int \frac{3x-2}{2\sqrt{x-1}}\,dx = x\sqrt{x-1}$

$\int \frac{9x-6}{4\sqrt{x-1}}\,dx = \frac{3}{2}\int \frac{3x-2}{2\sqrt{x-1}}\,dx$

$\frac{3}{2}\int \frac{3x-2}{2\sqrt{x-1}}\,dx = \frac{3}{2}\left(x\sqrt{x-1}\right)$

$\therefore \int \frac{9x-6}{4\sqrt{x-1}}\,dx = \frac{3}{2}\left(x\sqrt{x-1}\right)$

7 $y = -2x + 6 = 4x^2$

$4x^2 + 2x - 6 = 0$

$2x^2 + x - 3 = 0$

$(2x + 3)(x - 1) = 0$

$x = -\frac{3}{2},\ x = 1$

(a) Area of shaded region $= \int_0^1 6 - 2x\,dx - \int_0^1 4x^2\,dx$

$= \int_0^1 6 - 2x - 4x^2\,dx$

$= \left[6x - x^2 - \frac{4}{3}x^3\right]_0^1$

$= \left[6(1) - 1^2 - \frac{4}{3}(1)^3\right] - 0$

$= \frac{11}{3}$ units^2

(b) Generated volume $= \pi\int_0^4 \frac{y}{4}\,dy + \pi\int_4^6 \left(\frac{6-y}{2}\right)^2 dy$

$= \pi\left[\frac{y^2}{8}\right]_0^4 + \pi\left[\frac{1}{4}\cdot\frac{(6-y)^3}{-3}\right]_4^6$

$= \pi\left(\frac{4^2}{8}\right) + \pi\left[-\frac{(6-y)^3}{12}\right]_4^6$

$= 2\pi + \pi\left[0 - \left(-\frac{(6-4)^3}{12}\right)\right]$

$= 2\pi + \frac{2}{3}\pi$

$= \frac{8}{3}\pi$ units^3

Paper 2

1 (a) $\frac{d}{dx} = \left(x\sqrt{x^2+4}\right) = \frac{d}{dx}\left[x(x^2+4)^{\frac{1}{2}}\right]$

$= 1(x^2+4)^{\frac{1}{2}} + x\left(\frac{1}{2}\right)(x^2+4)^{\frac{1}{2}}(2x)$

$= \sqrt{x^2+4} + \frac{x^2}{\sqrt{x^2+4}}$

$= \frac{x^2 + 4 + x^2}{\sqrt{x^2+4}}$

$= \frac{2x^2 + 4}{\sqrt{x^2+4}}$ (Shown)

Observe that:

$\int \frac{2x^2+4}{\sqrt{x^2+4}}\,dx = x\sqrt{x^2+4}$

$2\int \frac{x^2+2}{\sqrt{x^2+4}}\,dx = x\sqrt{x^2+4}$

$\therefore \int \frac{x^2+2}{\sqrt{x^2+4}}\,dx = \frac{x}{2}\sqrt{x^2+4}$

Thus,

$$3\int_{-1}^{2} \frac{x^2+2}{\sqrt{x^2+4}}\,dx = 3\left[\frac{x}{2}\sqrt{x^2+4}\right]_{-1}^{2}$$
$$= 3\left[\frac{2}{2}\sqrt{4+4}\right] - \left(-\frac{1}{2}\right)\sqrt{1+4}\Big]$$
$$= 3\left(\sqrt{8} + \frac{1}{2}\sqrt{5}\right)$$
$$= 3\left(2\sqrt{2} + \frac{1}{2}\sqrt{5}\right)$$
$$= 6\sqrt{2} + \frac{3}{2}\sqrt{5}$$

(b) $y = x^3 - ax^{-1} + c$

$$\frac{dy}{dx} = 3x^2 + \frac{a}{x^2}$$
$$3x^2 + \frac{a}{x^2} = bx^2 + \frac{1}{x^2}$$
$$\therefore a = 1, b = 3$$
$$(2, 0),\ 0 = 2^3 - \frac{1}{2} + c$$
$$c = -\frac{15}{2}$$

2 (a) $\int_{-3}^{p} (5+x)^3\,dx = 320$

$$\left[\frac{(5+x)^4}{4}\right]_{-3}^{p} = 320$$
$$\frac{(5+p)^4}{4} - \frac{2^4}{4} = 320$$
$$\frac{(5+p)^4}{4} = 324$$
$$(5+p)^4 = 1\,296$$
$$= 6^4$$
$$5 + p = 6$$
$$p = 1$$

(b) $\frac{dy}{dx} = 3x^2 + ax + b$

(i) Stationary point $= (1, 0)$

$$\frac{dy}{dx} = 0$$
$$3(1)^2 + a(1) + b = 0$$
$$3 + a + b = 0$$
$$a + b = -3 \text{ ...... ①}$$
$$3(-3)^2 + a(-3) + b = 0$$
$$27 - 3a + b = 0$$
$$-3a + b = -27 \text{ ...... ②}$$

① − ②:

$$4a = 24$$
$$a = 6$$
$$b = -3 - a$$
$$= -3 - 6$$
$$= -9$$

(ii) $y = \int 3x^2 + 6x - 9\,dx$

$$= x^3 + 3x^2 - 9x + c$$
$$(1, 0),\ 0 = 1^3 + 3(1)^2 - 9(1) + c$$
$$0 = -5 + c$$
$$c = 5$$
$$\therefore y = x^3 + 3x^2 - 9x + 5$$

3 $\frac{dy}{dx} = a - \frac{b}{x^2}$

(a) At (1, 15), $\frac{dy}{dx} = -7$

$$a - \frac{b}{1^2} = -7$$
$$a - b = -7 \text{ ...... ①}$$

Given $\left(\frac{4}{3}, 14\right)$ is a turning point

$$\frac{dy}{dx} = 0$$
$$a - \frac{b}{\left(\frac{4}{3}\right)^2} = 0$$
$$a - \frac{9}{16}b = 0$$
$$a = \frac{9}{16}b \text{ .... ②}$$

Substitute ② into ①,

$$\frac{9}{16}b - b = -7$$
$$-\frac{7}{16}b = -7$$
$$b = 16$$
$$\therefore a = \frac{9}{16}(16) = 9$$

(b) $y = \int 9 - \frac{16}{x^2}dx$

$$= \int 9 - 16x^{-2}\,dx$$
$$= 9x - \frac{16x^{-1}}{-1} + c$$
$$= 9x + \frac{16}{x} + c$$
$$(1, 15),\ 15 = 9(1) + \frac{16}{1} + c$$
$$15 = 25 + c$$
$$c = -10$$
$$\therefore y = 9x + \frac{16}{x} - 10$$