FORM 5

CHAPTER 2 Differentiation

Self Test 1

1 (a) $\lim_{x\to 2} \frac{1-6x^2}{x+1} = \frac{1-6(2)^2}{2+1}$
$= \frac{1-6(4)}{3}$
$= -\frac{23}{3}$

(b) $\lim_{x\to 1} \frac{2x-2}{x^2+1} = \lim_{x\to 1} \frac{2(x-1)}{(x-1)(x+1)}$
$= \lim_{x\to 1} \frac{2}{x+1}$
$= \frac{2}{1+1}$
$= 1$

2 $y = 8x + 5$
$\frac{dy}{dx} = \lim_{\delta x\to 0} \frac{[8(x+\delta x)+5]-(8x+5)}{\delta x}$
$= \lim_{\delta x\to 0} \frac{8x+8\delta x+5-8x-5}{\delta x}$
$= \lim_{\delta x\to 0} 8$
$= 8$

(a) $y = \frac{2}{3x}$ .... ①
$y + \delta y = \frac{2}{3(x+\delta x)}$ .... ②
② – ①:
$\delta y = \frac{2}{3(x+\delta x)} - \frac{2}{3x}$
$= \frac{6x - 2(3x+3\delta x)}{3x(3x+3\delta x)}$
$= \frac{-6\delta x}{9x^2+9x\delta x}$
$\frac{\delta y}{\delta x} = \frac{-6}{9x^2+9x\delta x}$
$\frac{dy}{dx} = \lim_{\delta x\to 0} \frac{-6}{9x^2+9x\delta x}$
$= -\frac{6}{9x^2}$
$= -\frac{2}{3x^2}$

(b) $y = 2(1-x^2)$
$y = 2 - 2x^2$ .... ①
$y + \delta y = 2 - 2(x+\delta x)^2$
$= 2 - 2(x^2 + 2x\delta x + (\delta x)^2)$
$y + \delta y = 2 - 2x^2 - 4x\delta x - 2(\delta x)^2$ .... ②
② – ①:
$\delta y = -4x\delta x - 2(\delta x)^2$
$\frac{\delta y}{\delta x} = -4x - 2\delta x$
$\frac{dy}{dx} = \lim_{\delta x\to 0} -4x - 2\delta x$
$= -4x$

Self Test 2

1 (a) $y = 2x^4 + 3x^3 - 8x$
$\frac{dy}{dx} = 2(4x^3) + 3(3x^2) - 8$
$= 8x^3 + 9x^2 - 8$

(b) $y = 2(3x+1)(x-2)$
$= 2(3x^2 - 5x - 2)$
$= 6x^2 - 10x - 4$
$\frac{dy}{dx} = 12x - 10$

(c) $y = \frac{3x(1-x)}{2x-1}$
$= \frac{3x-3x^2}{2x-1}$
$\frac{dy}{dx} = \frac{(3-6x)(2x-1) - 2(3x-3x^2)}{(2x-1)^2}$
$= \frac{6x - 3 - 12x^2 + 6x - 6x + 6x^2}{(2x-1)^2}$
$= \frac{6x-3-6x^2}{(2x-1)^2}$
$= \frac{3(2x-2x^2-1)}{(2x-1)^2}$

(d) $y = (3x+4)^2(2x+3)$
$\frac{dy}{dx} = 2(3x+4)(3)(2x+3) + 2(3x+4)^2$
$= 6(3x+4)(2x+3) + 2(3x+4)^2$
$= 2(3x+4)[3(2x+3) + (3x+4)]$
$= 2(3x+4)(6x+9+3x+4)$
$= 2(3x+4)(9x+13)$

2 $f(x) = (x+3)^2 - 5(x+1)$
$f'(x) = 2(x+3)(1) - 5(1)$
$= 2x + 6 - 5$
$= 2x + 1$
$f'(1) = 2(1) + 1 = 3$

3 $f(t) = \frac{(t^2+6)^2}{t}, f'(t) = -21$
$f'(t) = \frac{2(t^2+6)(2t)(t) - 1(t^2+6)^2}{t^2}$
$= \frac{4t^2(t^2+6) - (t^2+6)^2}{t^2}$
$= \frac{(t^2+6)[4t^2 - (t^2+6)]}{t^2}$
$= \frac{(t^2+6)(3t^2-6)}{t^2}$
$= \frac{3t^4 + 12t^2 - 36}{t^2}$
$f'(t) = -21$
$3t^4 + 12t^2 - 36 = -21t^2$
$3t^4 + 33t^2 - 36 = 0$
$t^4 + 11t^2 - 12 = 0$
$(t^2-1)(t^2+12) = 0$
$t^2 = -12$ (undefined)
$\therefore t^2 = 1 \Rightarrow t = -1, t = 1$

4 $y = 3t - \frac{2}{t}$, $2x + t = 1$
$t = 1 - 2x$

(a) $\frac{dy}{dt}=3+\frac{2}{t^2}$, $\frac{dt}{dx}=-2$

$\frac{dy}{dx}=\frac{dy}{dt}\times\frac{dt}{dx}$

$=\left(3+\frac{2}{t^2}\right)(-2)$

$=-6-\frac{4}{t^2}$

$=-6-\frac{4}{(1-2x)^2}$

(b) $t=2,\ 1-2x=2$

$2x=-1$

$x=-\frac{1}{2}$

$\frac{dy}{dx}=-6-\frac{4}{\left[1-2\left(-\frac{1}{2}\right)\right]^2}$

$=-6-\frac{4}{2^2}$

$=-7$

Self Test 3

1 $s=\frac{2t^3-t}{3t^2}$

$=\frac{2t^3}{3t^2}-\frac{t}{3t^2}$

$=\frac{2}{3}t-\frac{1}{3t}$

$=\frac{2}{3}t-\frac{1}{3}t^{-1}$

$\frac{ds}{dt}=\frac{2}{3}+\frac{1}{3}t^{-2}$

$\frac{d^2s}{dt^2}=-\frac{2}{3}t^{-3}$

$=-\frac{2}{3t^3}$

2 $f(x)=\frac{3x^2}{2x+5}$

$f'(x)=\frac{6x(2x+5)-2(3x^2)}{(2x+5)^2}$

$=\frac{12x^2+30x-6x^2}{(2x+5)^2}$

$=\frac{6x^2+30x}{(2x+5)^2}$

$f''(x)=\frac{(12x+30)(2x+5)^2-2(2x+5)(2)(6x^2+30x)}{(2x+5)^4}$

$=\frac{(12x+30)(2x+5)^2-4(2x+5)(6x^2+30x)}{(2x+5)^4}$

$=\frac{(2x+5)\left[(12x+30)(2x+5)-4(6x^2+30x)\right]}{(2x+5)^4}$

$=\frac{(12x+30)(2x+5)-4(6x^2+30x)}{(2x+5)^3}$

$f''(-2)=\frac{(-24+30)(-4+5)-4(24-60)}{[2(-2)+5]^3}$

$=6-4(-36)$

$=150$

3 $y=-\frac{4}{x^2}+2x-1=-4x^{-2}+2x-1$

$\frac{dy}{dx}=8x^{-3}+2$

$\frac{d^2y}{dx^2}=-24x^{-4}=-\frac{24}{x^4}$

$x=2,\ \frac{d^2y}{dx^2}=-\frac{24}{2^4}$

$=-\frac{3}{2}$

Self Test 4

1 $f(x)=3x^2+5ax-12$

$m_{normal}=\frac{1}{3}\Rightarrow m_{tangent}=-3$

$f'(x)=6x+5a$

$f'(2)=6(2)+5a$

$-3=12+5a$

$5a=-15$

$a=-3$

$\therefore f(x)=3x^2-15x-12$

$f(2)=3(2)^2-15(2)-12$

$=-30$

Equation of tangent: $\frac{y+30}{x-2}=-3$

$y+30=-3x+6$

$y=-3x-24$

2 $y=ax^2+bx$

(a) $\frac{dy}{dx}=2ax+b$

$x=1,\ \frac{dy}{dx}=2a(1)+b$

$2=2a+b$ ...... ①

$(1,-3),\quad -3=a(1)^2+b(1)$

$a+b=-3$ ...... ②

① − ②: $a=5$

$b=-3-a=-8$

(b) $\frac{dy}{dx}=2(5)x+(-8)$

$=10x-8$

Stationary point: $\frac{dy}{dx}=0$

$10x-8=0$

$x=\frac{4}{5}$

$y=5x^2-8x$

$x=\frac{4}{5},\ y=5\left(\frac{4}{5}\right)^2-8\left(\frac{4}{5}\right)$

$=-\frac{16}{5}$

Thus, the stationary point is $\left(\frac{4}{5},-\frac{16}{5}\right)$

$\frac{d^2y}{dx^2}=10>0$

$\therefore \left(\frac{4}{5},-\frac{16}{5}\right)$ is a minimum point

3 $s=147t-4.9t^2$

(a) $\frac{ds}{dt}=147-9.8t$

Minimum point: $147-9.8t=0$

$9.8t=147$

$t=15$ seconds

(b) Maximum height,

$s=147(15)-4.9(15)^2$

$=1\,102.5$ metres

4 (a) $AC^2=AB^2+BC^2$

$y^2=x^2+1$

$y=(x^2+1)^{\frac{1}{2}}$

(b) $\frac{dy}{dx}=\frac{1}{2}(x^2+1)^{-\frac{1}{2}}(2x)$

$=\frac{x}{\sqrt{x^2+1}}$

$x = 6, \frac{dy}{dx} = \frac{6}{\sqrt{37}}$

$\frac{dy}{dt} = \frac{dy}{dx} \times \frac{dx}{dt}$

$= \frac{6}{\sqrt{37}} \times (-2)$

$= -1.97$ m per second

5 $y = \frac{4}{(x+2)^2} = 4(x+2)^{-2}$

$\frac{dy}{dx} = -8(x+2)^{-3}(1)$

$= -\frac{8}{(x+2)^3}$

$x = 1, \frac{dy}{dx} = -\frac{8}{(1+2)^3} = -\frac{8}{27}$

(a) $y = \frac{4}{(2.99)^2} = \frac{4}{(2+0.99)^2}$

$x_i = 0.99, x_0 = 1, \delta x = -0.01$

$\delta y \approx \frac{dy}{dx}(\delta x)$

$= -\frac{8}{27}(-0.01)$

$= \frac{0.08}{27}$

$y_0 = \frac{4}{(1+2)^2} = \frac{4}{9}$

$y_1 = \frac{4}{9} + \frac{0.08}{27}$

$= \frac{12.08}{27}$

$= \frac{1\ 208}{2\ 700}$

$= \frac{302}{675}$

(b) $y = \frac{4}{(3.03)^2} = \frac{4}{(2+1.03)^2}$

$x_i = 1.03, x_0 = 1, \delta x = 0.03$

$\delta y = -\frac{8}{27}(0.03)$

$= -\frac{0.24}{27}$

$y_1 = \frac{4}{9} + \left(-\frac{0.24}{27}\right)$

$= \frac{12 - 0.24}{27}$

$= \frac{11.76}{27}$

$= \frac{1\ 176}{2\ 700}$

$= \frac{98}{225}$

SPM Practice

Paper 1

1 $f(x) = \frac{5}{\sqrt{2x-1}} = 5(2x-1)^{-\frac{1}{2}}$

$f'(x) = -\frac{5}{2}(2x-1)^{-\frac{3}{2}}(2)$

$= -5(2x-1)^{-\frac{3}{2}}$

$f'(1) = -5(1)^{-\frac{3}{2}} = -5$

2 $s = 6x^2 + x, t = 3 - x$

$\frac{ds}{dx} = 12x + 1$

$x = 3 - t \Rightarrow \frac{dx}{dt} = -1$

$\frac{ds}{dt} = \frac{ds}{dx} \times \frac{dx}{dt}$

$= (12x+1)(-1)$

$= -12x - 1$

$= -12(3-t) - 1$

$= -36 + 12t - 1$

$= 12t - 37$

3 $y = (5-x)(2+x)^3$

(a) Gradient of the curve $= \frac{dy}{dx}$

$\frac{dy}{dx} = (-1)(2+x)^3 + (5-x)(3)(2+x)^2$

$= -(2+x)^3 + (15-3x)(2+x)^2$

$x = -1, \frac{dy}{dx} = -(2-1)^3 + (15+3)(2-1)^2$

$= 17$

(b) $m_{\text{tangent}} = 17, m_{\text{normal}} = -\frac{1}{17}$

$x = -1, y = (5+1)(2-1)^3 = 6$

Equation of normal: $\frac{y-6}{x+1} = -\frac{1}{17}$

$17y - 102 = -x - 1$

$17y + x = 101$

4 $y = \frac{1}{5}(6x - x^2)$

$= \frac{6}{5}x - \frac{1}{5}x^2$

$\frac{dy}{dx} = \frac{6}{5} - \frac{2}{5}x$

$2 = \frac{6}{5} - \frac{2}{5}x$

$-\frac{2}{5}x = \frac{4}{5}$

$x = -2$

$y = \frac{1}{5}(-12-4) = -\frac{16}{5}$

$\therefore A\left(-2, -\frac{16}{5}\right)$

5 (a) $f(x) = 2x^3 + ax^2 - 24x + 1$

$f'(x) = 0$

$6x^2 + 2ax - 24 = 0$

$f'(-4) = 0$

$6(-4)^2 + 2(-4)a - 24 = 0$

$8a = 72$

$a = 9$

(b) $y = 2(5x-2)^4$

$\frac{dy}{dx} = 8(5x-2)^3(5)$

$= 40(5x-2)^3$

$\frac{d^2y}{dx^2} = 120(5x-2)^2(5)$

$= 600(5x-2)^2$

6 $y = ax^2 + bx + c$

$\frac{dy}{dx} = 2ax + b$

$x = -2, \quad \frac{dy}{dx} = 2a(-2) + b$

$0 = -4a + b$

$b = 4a$ ...... ①

$(-1, 5),\ 5 = a(-1)^2 + b(-1) + c$
$5 = a - b + c$
$5 = a - 4a + c$
$3a = c - 5 \cdots\cdots ②$

$(-2, 2),\ 2 = a(-2)^2 + b(-2) + c$
$2 = 4a - 2b + c$
$2 = 4a - 8a + c$
$2 = c - 4a$
$c = 4a + 2 \cdots\cdots ③$

Substitute ③ into ②,
$3a = 2 + 4a - 5$
$a = 3$
$b = 4a = 12$
$c = 2 + 4(3) = 14$

7 (a) $pV = 500, \dfrac{dp}{dt} = -10, p = 20$

$V = 500p^{-1}$

$\dfrac{dV}{dp} = -500p^{-2}$

$= \dfrac{500}{p^2}$

$p = 20,\ \dfrac{dV}{dp} = -\dfrac{500}{(20)^2}$

$= -\dfrac{5}{4}$

$\dfrac{dV}{dt} = \dfrac{dV}{dp} \times \dfrac{dp}{dt}$

$= -\dfrac{5}{4}(-10)$

$= \dfrac{25}{2}$ units/s

(b) $A = \pi j^2$

$\dfrac{dA}{dj} = 2\pi j$

$j = 14,\ \dfrac{dA}{dj} = 2\pi(14) = 28\pi$

$\dfrac{dA}{dt} = \dfrac{dA}{dj} \times \dfrac{dj}{dt}$

$= 28\pi \times 0.55$

$= 15.4\pi$ cm^2/s

8 $y = (3x - 4)^2$

$\delta x = 2 - p - 2 = -p$

$\dfrac{dy}{dx} = 2(3x - 4)(3)$

$= 6(3x - 4)$

$x = 2,\ \dfrac{dy}{dx} = 6(6 - 4) = 12$

$\delta y \approx \dfrac{dy}{dx}(\delta x)$

$= 12(-p)$

$= -12p$

9 $y = \dfrac{1}{x^3 - 1} = (x^3 - 1)^{-1}$

$\dfrac{dy}{dx} = -1(x^3 - 1)^{-2}(3x^2)$

$= -\dfrac{3x^2}{(x^3 - 1)^2}$

$x = 2,\ \dfrac{dy}{dx} = -\dfrac{3(2)^2}{(2^3 - 1)^2}$

$= -\dfrac{12}{49}$

$y_0 = \dfrac{1}{2^3 - 1}$

$= \dfrac{1}{7}$

$\delta y \approx \dfrac{dy}{dx}(\delta x)$

$= -\dfrac{12}{49}(0.1)$

$= -\dfrac{12}{490}$

$= -\dfrac{6}{245}$

$\therefore \dfrac{1}{2.1^3 - 1} = \dfrac{1}{7} + \left(-\dfrac{6}{245}\right)$

$= \dfrac{29}{245}$

10 (a) $\displaystyle\lim_{x\to5} \frac{x - 5}{x^2 - 4x - 5} = \lim_{x\to5} \frac{x - 5}{(x - 5)(x + 1)}$

$\displaystyle = \lim_{x\to5} \frac{1}{(x + 1)}$

$= \dfrac{1}{6}$

(b) $\displaystyle\lim_{x\to\infty} \frac{2 - 5x^4}{3x^2 + 1} = \lim_{x\to\infty} \frac{\frac{2}{x^2} - \frac{5x^4}{x^2}}{\frac{3x^2}{x^2} + \frac{1}{x^2}}$

$\displaystyle = \lim_{x\to\infty} \frac{\frac{2}{x^2} - 5x^2}{3 + \frac{1}{x^2}}$

When $x \to \infty$, $\dfrac{2}{x^2} \to 0$, $\dfrac{1}{x^2} \to 0$, thus

$\displaystyle\lim_{x\to\infty} \frac{\frac{2}{x^2} - 5x^2}{3 + \frac{1}{x^2}} = -\infty$

Paper 2

1 (a) $y = -\dfrac{2}{3 - x} \cdots\cdots\cdots ①$

$y + \delta y = \dfrac{2}{3 - (x + \delta x)}$

$= -\dfrac{2}{3 - x - \delta x} \cdots\cdots\cdots ②$

② – ①:

$\delta y = -\dfrac{2}{3 - x - \delta x} + \dfrac{2}{3 - x}$

$= \dfrac{-2(3 - x) + 2(3 - x - \delta x)}{(3 - x)(3 - x - \delta x)}$

$= \dfrac{-6 + 2x + 6 - 2x - 2\delta x}{(3 - x)(3 - x - \delta x)}$

$\dfrac{\delta y}{\delta x} = \dfrac{-2\delta x}{(3 - x)(3 - x - \delta x)} \div \delta x$

$\displaystyle\frac{dy}{dx} = \lim_{\delta x\to0} \frac{-2}{(3 - x)(3 - x - \delta x)}$

$= -\dfrac{2}{(3 - x)^2}$

(b) $y = 2x(x - 3)^2$

(i) $\dfrac{dy}{dx} = 2(x - 3)^2 + 2x(2)(x - 3)(1)$

$= 2(x - 3)^2 + 4x(x - 3)$

$= (x - 3)[2(x - 3) + 4x]$

$= (x - 3)(6x - 6)$

$= 6(x - 3)(x - 1)$

(ii) Gradient of the curve $= -6$

$6(x-3)(x-1) = -6$
$(x-3)(x-1) = -1$
$x^2 - 4x + 3 + 1 = 0$
$x^2 - 4x + 4 = 0$
$(x-2)^2 = 0$
$x = 2$

(c) $y = -6x + 1 \Rightarrow m = -6$

$y = x^3 + ax^2 - 5x + 1$

$\frac{dy}{dx} = 3x^2 + 2ax - 5$

$-6 = 3x^2 + 2ax - 5$

$x = -1,\ 3(-1)^2 + 2a(-1) - 5 = -6$
$-2a - 2 = -6$
$2a = 4$
$a = 2$

$x = -1, y = (-1)^3 + 2(-1)^2 - 5(-1) + 1$
$= 7$

Equation of tangent: $\frac{y-7}{x+1} = -6$
$y - 7 = -6x - 6$
$y + 6x = 1$

2 (a) $y = 2x^2 + (x+2)^{-2}$

(i) $\frac{dy}{dx} = 4x + (-2)(x+2)^{-3}$
$= 4x - \frac{2}{(x+2)^3}$

$x = -3, \frac{dy}{dx} = 4(-3) + \frac{2}{(-3+2)^3}$
$= -10$

(ii) $m_{normal} = \frac{1}{10}$

$x = -3, y = 2(-3)^2 + \frac{1}{(-3+2)^2}$
$= 19$

Equation of normal: $\frac{y-19}{x+3} = \frac{1}{10}$
$10y - 190 = x + 3$
$10y - x = 193$

(b) $V = \frac{1}{3}\pi x^2(21 - x), \frac{dV}{dt} = 2, x = 2$

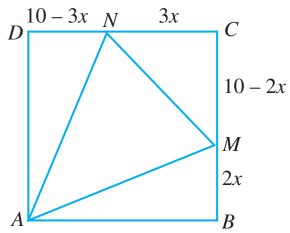
$V = 7\pi x^2 - \frac{1}{3}\pi x^3$

$\frac{dV}{dx} = 14\pi x - \pi x^2$

$x = 2, \frac{dV}{dx} = 14(2)\pi - \pi(2)^2$
$= 24\pi$

$\frac{dx}{dt} = \frac{dx}{dV} \times \frac{dV}{dt}$
$= \frac{1}{24\pi} \times 2$
$= \frac{1}{12\pi}$ cm/s

3 (a)



(i) Area of $\triangle AMN$

$= 10(10) - \frac{1}{2}(10-3x)(10) - \frac{1}{2}(2x)(10) - \frac{1}{2}(3x)(10-2x)$
$= 100 - (50 - 15x) - 10x - 3x(5 - x)$
$= 100 - 50 + 15x - 10x - 15x + 3x^2$
$= (50 - 10x + 3x^2)$ cm^2 (Shown)

(ii) Area is minimum, thus $\frac{dA}{dx} = 0$

$\frac{dA}{dx} = 6x - 10$

$6x - 10 = 0$

$x = \frac{5}{3}$

(iii) $\frac{dA^2}{dx^2} = 6 > 0$

$\therefore A = 50 - 10\left(\frac{5}{3}\right) + 3\left(\frac{5}{3}\right)^2$
$= \frac{125}{3}$ cm^2

(b) (i) $y = 5 - 3s^2,$ $s = 5x + 1$

$\frac{dy}{ds} = -6s$ $\frac{ds}{dx} = 5$

$\frac{dy}{dx} = \frac{dy}{ds} \times \frac{ds}{dx}$
$= -30s$
$= -30(5x+1)$
$= -150x - 30$

(ii) $\delta x = 3.98 - 4 = -0.02$

(a) $\frac{ds}{dx} = 5$

$\delta s \approx \frac{ds}{dx}(\delta x)$
$= 5(-0.02)$
$= -0.1$

(b) $\delta y \approx \frac{dy}{dx}(\delta x)$
$= [-150(4) - 30](-0.02)$
$= (-630)(-0.02)$
$= 12.6$