Fully-Worked Solutions

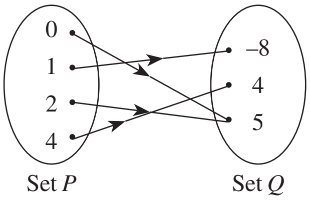
FORM 4

CHAPTER 1 Functions

Self Test 1

1 $\{(2, 5), (1, -8), (0, 5), (4, 4)\}$

(a)



(b) Type of relation: Many-to-one
The objects (0 and 2) are mapped to the same image, that is 5.

(c) Domain = $\{0, 1, 2, 4\}$
Range = $\{-8, 4, 5\}$

(d) Object of 5 = 0, 2

2 (a) $h(1-x) = 8 + 3x$
Let $1 - x = y$
$x = 1 - y$
$h(y) = 8 + 3(1 - y)$
$= 8 + 3 - 3y$
$= 11 - 3y$
$\therefore h(x) = 11 - 3x$

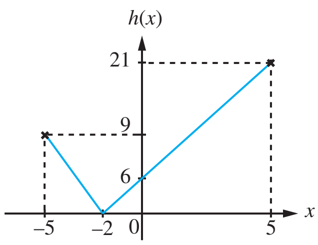
(b) $1 - x = 5$
$x = 1 - 5 = -4$
$h(-4) = 11 - 3(-4)$
$= 11 + 12$
$= 23$
$\therefore a = 23$

Alternative method
$h(1-x) = 8 + 3x$
$h(5) = 8 + 3(5)$
$a = 23$

$h(1-x) = 8 + 3x$
$h(b) = 8 + 3b$
$2 = 8 + 3b$
$3b = -6$
$b = -2$

(c) $h(x) = 11 - 3x$
$h(x) = x$
$11 - 3x = x$
$11 = 4x$
$x = \frac{11}{4}$

3 $h(x) = |6 + 3x|, -5 \leqslant x \leqslant 5$
$x = 0, h(0) = |6 + 0| = 6$
$x = -5, h(-5) = |6 + 3(-5)| = 9$
$x = 5, h(5) = |6 + 3(5)| = 21$
$h(x) = 0 \Rightarrow 6 + 3x = 0$
$x = -2$



(a) Type of relation: Many-to-one
(b) Range: $0 \leqslant h(x) \leqslant 21$
(c) $x = 2, h(2) = |6 + 3(2)| = 12$
(d) $h(x) = 1$
$|6 + 3x| = 1$
$6 + 3x = 1$ or $6 + 3x = -1$
$3x = -5$ $\quad$ $3x = -7$
$x = -\frac{5}{3}$ $\quad$ $x = -\frac{7}{4}$

Self Test 2

1 $f(x) = px - 3, g(x) = 5x + 7$
(a) $fg(x) = f(5x + 7)$
$= p(5x + 7) - 3$
$= 5px + (7p - 3)$
$gf(x) = g(px - 3)$
$= 5(px - 3) + 7$
$= 5px - 8$
$fg = gf$
$5px + 7p - 3 = 5px - 8$
$7p = -5$
$p = -\frac{5}{7}$

(b) $fg(x) = 7$
$f(5x + 7) = 7$
$-\frac{5}{7}(5x + 7) - 3 = 7$
$-\frac{5}{7}(5x + 7) = 10$
$5x + 7 = -14$
$x = -\frac{21}{5}$

2 $f(x) = \frac{b}{x - 1}, g(x) = 6 - ax$
(a) $g(2) = 2$
$6 - a(2) = 2$
$2a = 4$
$a = 2$

(b) $f(2) = \frac{b}{2 - 1} = b$
$g^2(0) = gg(0)$
$= g(6)$
$= 6 - 2(6)$
$= -6$
$f(2) = g^2(0)$
$\therefore b = -6$

(c) $gf(-3) = g\left(\frac{-6}{-3 - 1}\right)$
$= g\left(\frac{3}{2}\right)$
$= 6 - 2\left(\frac{3}{2}\right)$
$= 3$

3 $f(x) = 1 - 2x, g(x) = 2 - 3x$
(a) $fg(x) = f(2 - 3x)$
$= 1 - 2(2 - 3x)$
$= 1 - 4 + 6x$
$= 6x - 3$

$gf(x) = g(1 - 2x)$
$= 2 - 3(1 - 2x)$
$= 2 - 3 + 6x$
$= 6x - 1$

(b) $fg(x) = g(x + 2)$
$6x - 3 = 2 - 3(x + 2)$

$$6x - 3 = 2 - 3x - 6$$
$$9x = -1$$
$$x = -\frac{1}{9}$$

Self Test 3

1 $f(x) = \frac{6}{x} - \frac{1}{2}, x \neq p, g(x) = \frac{12}{2x + 1}, x \neq k$

(a) $x \neq 0 \Rightarrow p = 0$

$2x + 1 \neq 0 \Rightarrow k = -\frac{1}{2}$

(b) Let $f^{-1}(x) = y$

$$f(y) = x$$
$$\frac{6}{y} - \frac{1}{2} = x$$
$$\frac{6}{y} = x + \frac{1}{2}$$
$$6 = y\left(\frac{2x + 1}{2}\right)$$
$$y = \frac{12}{2x + 1}$$
$$\therefore f^{-1}(x) = \frac{12}{2x + 1}, x \neq -\frac{1}{2}$$

(c) $fg(x) = f\left(\frac{12}{2x + 1}\right)$

$$= \frac{6}{\frac{12}{2x + 1}} - \frac{1}{2}$$
$$= \frac{6(2x + 1)}{12} - \frac{1}{2}$$
$$= \frac{2x + 1}{2} - \frac{1}{2}$$
$$= \frac{2x}{2}$$
$$= x$$

$g(x)$ is an inverse function of $f(x)$.

2 $f(x) = 3x + a$, $ff(6) = 10$

(a)
$$ff(6) = f(3(6) + a)$$
$$10 = f(18 + a)$$
$$3(18 + a) + a = 10$$
$$54 + 4a = 10$$
$$a = -11$$

(b) Let $f^{-1}(3) = y$

$$f(y) = 3$$
$$3y - 11 = 3$$
$$3y = 14$$
$$y = \frac{14}{3}$$
$$\therefore f^{-1}(3) = \frac{14}{3}$$

3 (a) $g : -3 \rightarrow -2$

$\therefore g^{-1}(-2) = -3$

(b) $hg(8) = h(0)$

$= 5$

(c) $g^{-1}h^{-1}(-3)$

Observe that $h : -2 \rightarrow -3$

$\therefore h^{-1}(-3) = -2$

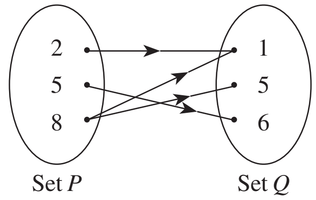
$g^{-1}(h^{-1}(-3)) = g^{-1}(-2)$

$= -3$

SPM Practice

Paper 1

1 (a) $\{(2, 1), (5, 6), (8, 1), (8, 5)\}$



(b) Type of relation: Many-to-many

(c) Domain = $\{2, 5, 8\}$

Range = $\{1, 5, 6\}$

(d) Object of 5 = 8

2 $h(t) = \frac{2t^2 - 3}{5}$

(a) $h(-2) = \frac{2(-2)^2 - 3}{5}$

$= 1$

(b) Object = 3 $\Rightarrow x = 3$

$$h(3) = \frac{2(3)^2 - 3}{5}$$
$$= \frac{15}{5}$$
$$= 3$$

(c)
$$h(t) = t$$
$$\frac{2t^2 - 3}{5} = t$$
$$2t^2 - 3 = 5t$$
$$2t^2 - 5t - 3 = 0$$
$$(2t + 1)(t - 3) = 0$$
$$t = -\frac{1}{2}, t = 3$$

3 $f(x) = \frac{3 - x}{2}$

(a)
$$f(a) = 4$$
$$\frac{3 - a}{2} = 4$$
$$3 - a = 8$$
$$a = -5$$

$$f(-3) = b$$
$$\frac{3 - (-3)}{2} = b$$
$$b = 3$$

(b)
$$f(x) = -12$$
$$\frac{3 - x}{2} = -12$$
$$3 - x = -24$$
$$x = 27$$

4 $f(x) = |6x - 4|$

(a) $x = 2, f(2) = |6(2) - 4| = 8$

(b) $f(-5) = |6(-5) - 4|$

$= |-34|$

$= 34$

(c) $f(x) = 8$

$|6x - 4| = 8$

$6x - 4 = 8$ or $6x - 4 = -8$

$6x = 12$ $\quad$ $6x = -4$

$x = 2$ $\quad$ $x = -\frac{2}{3}$

5 $f(x) = |x - 2| + a, -5 \leqslant x \leqslant n$

(a) $f(-5) = m$

$|-5 - 2| + a = m$

$7 + a = m \ldots$ ①

$$f(0) = 1$$
$$|0-2| + a = 1$$
$$2 + a = 1$$
$$a = -1$$

Substitute $a = -1$ into ①,

$$7 + (-1) = m$$
$$m = 6$$

$$f(n) = 3$$
$$|x-2| - 1 = 3$$
$$|x-2| = 4$$

$x - 2 = 4$ or $x - 2 = -4$ $(n > 0)$

$x = 6$ (not acceptable)

$\therefore n = 6$

(b) Range: $-1 \leqslant f(x) \leqslant 6$

6 $f(x) = 8x - 3$, $g(x) = 2x$

(a) $f(x) = 9$

$$8x - 3 = 9$$
$$8x = 12$$
$$x = \frac{3}{2}$$

(b) $f: g(x) \to 15$

$$fg(x) = 15$$
$$f(2x) = 15$$
$$8(2x) - 3 = 15$$
$$16x = 18$$
$$x = \frac{9}{8}$$

7 (a) $f(x) = x + 1$

$$fg(x) = 5x - 3$$
$$g(x) + 1 = 5x - 3$$
$$g(x) = 5x - 4$$

(b) $f(x) = ax + b$

$$ff(x) = 16x - 15$$
$$f(ax + b) = 16x - 15$$
$$a(ax + b) + b = 16x - 15$$
$$a^2x + ab + b = 16x - 15$$

$a^2 = 16$ $\quad$ $ab + b = -15$

$a = 4$ $\quad$ $4b + b = -15$

$5b = -15$

$b = -3$

8 $f(x) = ax + 3$, $g(x) = 12x - 7$

$$f(2) = a(2) + 3$$
$$= 2a + 3$$

Let $g^{-1}(5) = y$

$$g(y) = 5$$
$$12y - 7 = 5$$
$$12y = 12$$
$$y = 1$$

$\therefore g^{-1}(5) = 1$

$$f(2) = g^{-1}(5)$$
$$2a + 3 = 1$$
$$2a = -2$$
$$a = -1$$

9 $m(x) = 2x + 5$

(a) Let $m^{-1}(x) = y$

$$2y + 5 = x$$
$$2y = x - 5$$
$$y = \frac{x-5}{2}$$

$\therefore$ The function that maps Q to P is $m^{-1}(x) = \frac{x-5}{2}$

(b) $n(2x + 5) = 9 - 4x$

Let $2x + 5 = z$

$$x = \frac{z-5}{2}$$
$$n(z) = 9 - 4\left(\frac{z-5}{2}\right)$$
$$= 9 - 2z + 10$$
$$= 19 - 2z$$

$\therefore n(x) = 19 - 2x$

Paper 2

1 (a) $f^{-1}(x) = \frac{a}{x} + 3$

$$f(2) = 5 \Rightarrow f^{-1}(5) = 2$$
$$\frac{a}{5} + 3 = 2$$
$$\frac{a}{5} = -1$$
$$a = -5$$

(b) $f(x) = \frac{12}{4x-1}, x \neq p$

(i) $4x - 1 \neq 0$

$$x \neq \frac{1}{4}$$
$$\therefore p = \frac{1}{4}$$

(ii) $f(-3) = \frac{12}{4(-3)-1}$

$$= -\frac{12}{13}$$
$$f(6) = \frac{12}{4(6)-1}$$
$$= \frac{12}{23}$$

(iii) $f(x) = 4$

$$\frac{12}{4x-1} = 4$$
$$4x - 1 = 3$$
$$x = 1$$

2 (a) $fh(x) = f\left(\frac{3}{2x}\right)$

$$= -4\left(\frac{3}{2x}\right)$$
$$= -\frac{6}{x}, x \neq 0$$

(b) $gf(x) = g(-4x)$

$$= -4x + 8$$

$$fh(x) = gf(x)$$
$$-\frac{6}{x} = -4x + 8$$
$$-6 = -4x^2 + 8x$$
$$4x^2 - 8x - 6 = 0$$
$$2x^2 - 4x - 3 = 0$$
$$x = \frac{4 \pm \sqrt{(-4)^2 - 4(2)(-3)}}{2(2)}$$
$$= \frac{4 \pm \sqrt{40}}{4}$$
$$= 1 \pm \frac{\sqrt{10}}{2}$$

3 $f(x) = 8x - 1$, $g(x) = (x - 1)^2$

(a)
$$\begin{aligned} 3f(1) + 4 &= f(2k) \\ 3[8(1) - 1] + 4 &= 8(2k) - 1 \\ 3(7) + 4 &= 16k - 1 \\ 25 &= 16k - 1 \\ k &= \frac{26}{16} \\ &= \frac{13}{8} \end{aligned}$$

(b) (i)
$$\begin{aligned} gf(2) &= g[8(2) - 1] \\ &= g(15) \\ &= (15 - 1)^2 \\ &= 196 \end{aligned}$$

(ii) Let $f^{-1}(6) = y$
$$\begin{aligned} f(y) &= 6 \\ 8y - 1 &= 6 \\ y &= \frac{7}{8} \\ \therefore f^{-1}(6) &= \frac{7}{8} \end{aligned}$$

(c) Range: $gf(x) \geqslant 0$, $fg(x) \geqslant -1$